CONTENTS

LEARNING OBJECTIVES

After completing this lecture, students should be able to:

1. Explain the concept of **time-varying magnetic fields** and their distinction from static magnetic fields.
2. Apply **Faraday's law of electromagnetic induction** to determine the induced electromotive force (emf) in various configurations.
3. Differentiate between **transformer emf** and **motional emf**, and describe their physical origins.
4. Analyze the relationship between **electric and magnetic fields** in time-varying situations using **Maxwell's equations**.
5. Define and calculate **displacement current** and explain its role in ensuring current continuity.
6. Solve practical problems involving **induced voltage** and **current** in stationary and moving loops under changing magnetic fields

1 INTRODUCTION

In previous lectures, we studied static or steady magnetic fields, where the magnetic flux density **B** remains constant with time. However, in many practical situations — such as electric generators, transformers, and communication systems — magnetic fields vary with time. These **time-varying magnetic fields** induce electric fields that give rise to **electromotive forces (emf)**, leading to the generation of electric currents.

This phenomenon is described by **Faraday's law of electromagnetic induction**, which forms one of the fundamental building blocks of electromagnetic theory. Time-varying fields link electricity and magnetism dynamically, resulting in energy conversion and propagation of electromagnetic waves.

In this lecture, we will explore the mathematical representation and physical interpretation of time-varying magnetic fields, derive key relationships from **Faraday's law**, and examine special cases such as transformer emf, motional emf, and the combined effect of both. We will also introduce the concept of **displacement current**, which extends Ampère's law to time-varying conditions and completes **Maxwell's equations** for dynamic electromagnetic systems.

2 ELECTRIC POTENTIAL

Suppose we wish to move a point charge Q from point A to point B in an electric field **E** as shown in Figure 1. From Coulomb's law, the force on Q is $\mathbf{F} = Q\mathbf{E}$ so that the work done in displacing the charge by $d\mathbf{l}$ is:

$$dW = -\boldsymbol{F} \cdot d\boldsymbol{l} = -Q\boldsymbol{E} \cdot d\boldsymbol{l} \tag{1}$$

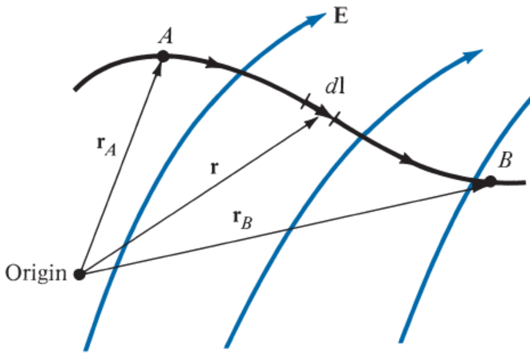


Figure 1: Displacement of point charge Q in an electrostatic field E.

The negative sign indicates that the work is being done by an external agent. Thus, the total work done, or the potential energy required, in moving Q from A to B, is

$$W = -Q \int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \tag{2}$$

Dividing W by Q gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the potential difference between points A and B. Thus

$$V_{AB} = \frac{W}{Q} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \tag{3}$$

Note that

1. In determining V_{AB}, A is the initial point while B is the final point.
2. If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement; an external agent performs the work.
3. V_{AB} is independent of the path taken (to be shown a little later).
4. V_{AB} is measured in joules per coulomb, commonly referred to as volts (V).

3 FARADAY'S LAW

According to Faradays experiments, a static magnetic field produces no current flow; but in a closed circuit, a time-varying field produces an induced voltage (called *electromotive force* or simply emf) that causes a flow of current.

Faraday discovered that the **induced emf, V_{emf}** (in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

This is called Faraday's law, and it can be expressed as:

$$V_{\mathrm{emf}} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \tag{4}$$

where $\lambda = N\Psi$ is the flux linkage, N is the number of turns in the circuit, and Ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This behavior is described as Lenz's law.

4 MOTIONAL ELECTROMOTIVE FORCES

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn N=1;

$$V_{emf} = -\frac{d\Psi}{dt} \tag{5}$$

In terms of **E** and **B**,

$$V_{emf} = \oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{S} \tag{6}$$

Where Ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by the closed path L. It is clear from Equation (6) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that $d\mathbf{l}$ and $d\mathbf{S}$ in

Equation (6) are in accordance with the right-hand rule as well as Stokes's theorem. The variation of flux with time may be caused in three ways:

- By having a stationary loop in a time-varying **B** field
- By having a time-varying loop area in a static **B** field
- By having a time-varying loop area in a time-varying **B** field

4.1 STATIONARY LOOP IN TIME-VARYING B FIELD (TRANSFORMER EMF)

In Figure 2 a stationary conducting loop is in a time-varying magnetic **B** field. Equation (6) becomes

$$V_{emf} = \oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} \qquad (7)$$

This emf induced by the time-varying current (producing the time-varying **B** field) in a stationary loop is often referred to as *transformer emf* in power analysis, since it is due to transformer action. By applying Stokes's theorem, we obtain:

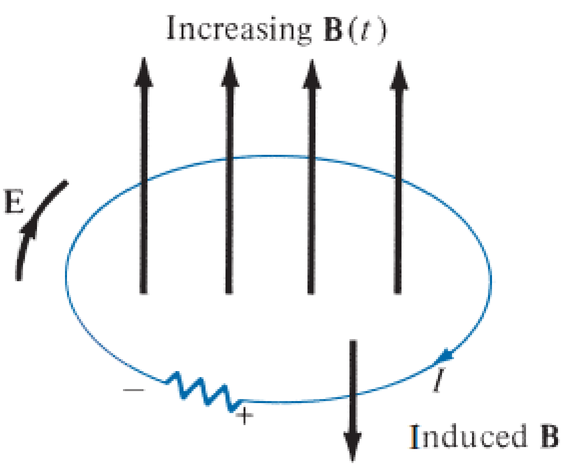


Figure 2: Induced emf due to a stationary loop in a time varying B field.

$$V_{emf} = \int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} \qquad (8)$$

that is,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (9)$$

This is one of the Maxwell's equations for time-varying fields.

4.2 MOVING LOOP IN STATIC B FIELD (MOTIONAL EMF)

When a conducting loop is moving in a static **B** field, an emf is induced in the loop. The force on a charge moving with uniform velocity **u** in a magnetic field **B** is:

$$\boldsymbol{F}_m = Q\boldsymbol{u} \times \boldsymbol{B} \qquad (10)$$

We define the motional electric field $\mathbf{E}_m$ as:

$$\boldsymbol{E}_m = \frac{\boldsymbol{F}_m}{Q} = \boldsymbol{u} \times \boldsymbol{B} \qquad (11)$$

If we consider a conducting loop, moving with uniform velocity **u** as consisting of a large number of free electrons, the emf induced in the loop is:

$$V_{emf} = \oint_L \boldsymbol{E}_m \cdot d\boldsymbol{l} = \oint_L (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \qquad (12)$$

This type of emf is called motional emf or flux-cutting emf because it is due to motional action. By applying Stokes's theorem, we obtain

$$V_{emf} = \int_S (\nabla \times \boldsymbol{E}_m) \cdot d\boldsymbol{S} = \int_S \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{S} \qquad (13)$$

To apply Equation (12) is not always easy; some care must be exercised. The following points should be noted.

1. The integral in Equation (12) is zero along the portion of the loop where $\mathbf{u} = \mathbf{0}$. Thus $d\mathbf{l}$ is taken along the portion of the loop that is cutting the field where $\mathbf{u}$ has nonzero value.

2. The direction of the induced current is the same as that of $\mathbf{E}_m$ or $\mathbf{u} \times \mathbf{B}$. The limits of the integral in Equation (12) are selected in the direction opposite to the induced current, thereby satisfying Lenz's law.

4.3 MOVING LOOP IN TIME-VARYING FIELD

In the general case, a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining Equation (7) and Equation (12) gives the total emf as:

$$V_{emf} = \oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} + \oint_L (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \qquad (14)$$

Thus,

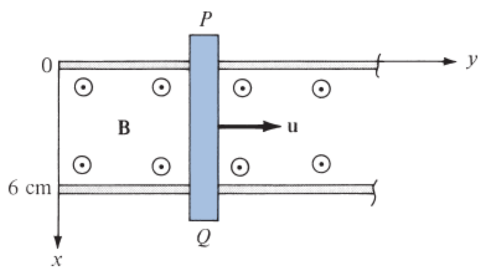
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \qquad (15)$$

EXAMPLE 1 : A conducting bar can slide freely over two conducting rails as shown below .Calculate the induced voltage in the bar:

(a) If the bar is stationed at $y = 8$ cm and $\mathbf{B} = 4\cos\ 10^6 t\mathbf{a}_z \text{mWb/m}^2$

(b) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\mathbf{a}_z \text{mWb/m}^2$

(c) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\cos\ (10^6 t - y)\mathbf{a}_z \text{mWb/m}^2$



Solution

(a) In this case, we have transformer emf given by

$$V_{\text{emf}} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^6)\sin\, 10^6 t\, dx dy$$
$$= 4(10^3)(0.08)(0.06)\sin\, 10^6 t$$
$$= 19.2\sin\, 10^6 t \text{ V}$$

(b) This is the case of motional emf:

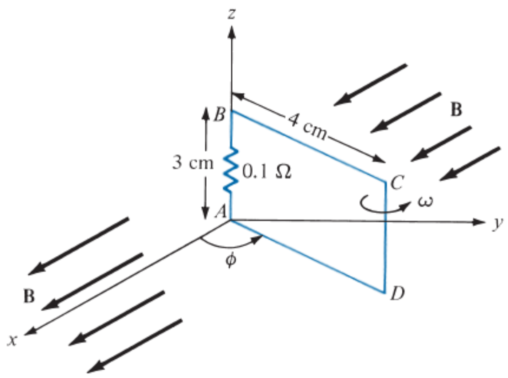
$$V_{\text{emf}} = \int_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{x=\ell}^{0} (u\mathbf{a}_y \times B\mathbf{a}_z) \cdot dx\mathbf{a}_x$$
$$= -uB\ell = -20(4 \times 10^{-3})(0.06)$$
$$= -4.8\text{mV}$$

(c)
$$V_{\text{emf}} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$
$$= \int_{x=0}^{0.06} \int_0^{y} 4.10^{-3}(10^6)\sin\,(10^6 t - y')dy'dx$$
$$+ \int_{0.06}^{0} \left[20\mathbf{a}_y \times 4.10^{-3}\cos\,(10^6 t - y)\mathbf{a}_z\right] \cdot dx\mathbf{a}_x$$
$$= 240\cos\,(10^6 t - y')\big|_0^y - 80(10^{-3})(0.06)\cos\,(10^6 t - y)$$
$$= 240\cos\,(10^6 t - y) - 240\cos\, 10^6 t - 4.8(10^{-3})\cos\,(10^6 t - y)$$
$$\simeq 240\cos\,(10^6 t - y) - 240\cos\, 10^6 t$$

EXAMPLE 2 : The loop shown below is inside a uniform magnetic field $\mathbf{B} = 50\mathbf{a}_x \text{mWb/m}^2$. If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop lies in the yz-plane at time $t = 0$, find

(a) The induced emf at $t = 1$ ms

(b) The induced current at $t = 3$ ms



Solution

(a) Since the **B** field is time invariant, the induced emf is motional, that is,

$$V_{\text{emf}} = \int_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

Where,

$$d\mathbf{l} = d\mathbf{l}_{DC} = dz\mathbf{a}_z, \mathbf{u} = \frac{d\mathbf{l}'}{dt} = \frac{\boldsymbol{\rho} d\boldsymbol{\phi}}{dt}\mathbf{a}_\phi = \boldsymbol{\rho}\omega\mathbf{a}_\phi$$
$$\boldsymbol{\rho} = AD = 4 \text{ cm}, \boldsymbol{\omega} = 2\pi f = 100\pi$$

Because **u** and $d\mathbf{l}$ are in cylindrical coordinates, we transform B into cylindrical coordinates

$$\mathbf{B} = B_o\mathbf{a}_x = B_o(\cos\,\phi\mathbf{a}_\rho - \sin\,\phi\mathbf{a}_\phi)$$

where $B_o = 0.05$. Hence,

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\omega & 0 \\ B_o\cos\,\phi & -B_o\sin\,\phi & 0 \end{vmatrix} = -\rho\omega B_o\cos\,\phi\mathbf{a}_z$$

$$(\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\rho\omega B_o\cos\,\phi dz = -0.04(100\pi)(0.05)\cos\,\phi dz$$
$$= -0.2\pi\cos\,\phi dz$$

$$V_{\text{emf}} = \int_{z=0}^{0.03} -0.2\pi\cos\,\phi dz = -6\pi\cos\,\phi\text{mV}$$

To determine ϕ, recall that

$$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + C_o$$

where C_o is an integration constant. At $t = 0, \phi = \pi/2$ because the loop is in the yz-plane at that time, $C_o = \pi/2$. Hence,

$$\phi = \omega t + \frac{\pi}{2}$$

and

$$V_{\text{emf}} = -6\pi\cos\left(\omega t + \frac{\pi}{2}\right) = 6\pi\sin\,(100\pi t)\text{mV}$$
$$\text{At } t = 1 \text{ ms}, V_{\text{emf}} = 6\pi\sin\,(0.1\pi) = 5.825\text{mV}$$

(b) The current induced is

$$i = \frac{V_{\text{emf}}}{R} = 60\pi\sin\,(100\pi t)\text{mA}$$

At $t = 3$ ms,

$$i = 60\pi\sin\,(0.3\pi)\text{mA} = 0.1525 \text{ A}$$

5 DISPLACEMENT CURRENT AND CONDUCTION CURRENT

We shall now reconsider Maxwell's curl equation for magnetic fields (Ampère's circuit law) for time-varying conditions.

For static EM fields, we recall that

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{16}$$

But the divergence of the curl of any vector field is identically zero. Hence,

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = 0 = \nabla \cdot \boldsymbol{J} \tag{17}$$

The continuity of current, requires that:

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \boldsymbol{\rho}_v}{\partial t} \neq 0 \tag{18}$$

Thus, the above equations are obviously incompatible for time-varying conditions. So, we add a term to (16) so that it becomes

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_d \tag{19}$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = 0 = \nabla \cdot \boldsymbol{J} + \nabla \cdot \boldsymbol{J}_d \tag{20}$$

Evaluating $\nabla \cdot \mathbf{J}$:

$$\nabla \cdot \boldsymbol{J}_d = -\nabla \cdot \boldsymbol{J} = \frac{\partial \boldsymbol{\rho}_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{D}) = \nabla \cdot \frac{\partial \boldsymbol{D}}{\partial t} \tag{21}$$

Or

$$\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{dt} \tag{22}$$

The final results we get:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad (23)$$

This is Maxwell's equation (based on Ampère's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial\mathbf{D}/\partial t$ is known as displacement current density and $\mathbf{J}$ is the conduction current. Based on the displacement current density, we define the displacement current as:

$$I_d = \int_S \boldsymbol{J}_d \cdot d\boldsymbol{S} + \int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot d\boldsymbol{S} \qquad (24)$$

EXAMPLE 3 : In free space, $\mathbf{E} = 20\cos(\omega t - 50x)\mathbf{a}_y$ V/m. Calculate

(a) J_d

(b) H

(c) ω

Solution

(a)

$$\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{\partial t} = -20\omega\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_y A/m^2$$

(b)

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_d \rightarrow -\frac{\partial H_z}{\partial x}\boldsymbol{a}_y = -20\omega\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_y$$

$$\text{or } \boldsymbol{H} = \frac{20\omega\varepsilon_o}{50}\cos(\omega t - 50x)\boldsymbol{a}_z$$

$$= 0.4\omega\varepsilon_o \cos(\omega t - 50x)\boldsymbol{a}_z \text{ A/m}$$

(c)

$$\nabla \times \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} \rightarrow \frac{\partial E_y}{\partial x}\boldsymbol{a}_z = 0.4\mu_o\omega^2\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_z$$

$$1000 = 0.4\mu_o\varepsilon_o\omega^2 = 0.4\frac{\omega^2}{c^2}$$

$$\text{or } \omega = 1.5 \times 10^{10} \text{rad/s}$$

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