1 Laplace Transforms

1.1 Definition

Let a function $f(t)$ be continuous and defined for a positive value of t The Laplace transform of $f(t)$ associate a function s defined by $\phi(s) = \int_0^\infty e^{-st} f(t)\, dt$

Here $\phi(s)$ is said to be the Laplace transform of $f(t)$ and it is denoted by $L(f(t))$, *or* $L(f)$ that is $L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

1. Find the Laplace transform of $ft) = \begin{cases} e^t, & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

   Ans.

$$\begin{aligned}
L(f(t)) &= \int_0^\infty e^{-st}\, dt \\
&= \int_0^1 e^{-st} e^t\, dt + \int_1^\infty e^{-st} 0\, dt \\
&= \int_0^1 e^{-(s-1)t}\, dt = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 \\
&= \frac{-(s-1)}{-(s-1)} - \frac{1}{-(s-1)} = \frac{e^{(1-s)}}{1-s} - \frac{1}{1-s}
\end{aligned}$$

2. Find the Laplace transform of $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

   Ans.

$$\begin{aligned}
L(f(t)) &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^{2\pi} e^{-st} \cos t\, dt \\
&= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{2\pi} \\
&= \frac{e^{-2\pi s}}{s^2+1}(-s) - \frac{1}{s^2+1}(-s) \\
&= (1 - e^{-2\pi s})\frac{s}{s^2+1}
\end{aligned}$$

Some basic Laplace Formulas

1. $L(1) = \frac{1}{s}$
2. $L(t) = \frac{1}{s^2}$
3. $L(t^n) = \frac{n!}{s^{n+1}}$ where n is an integer
4. $L(e^{at}) = \frac{1}{s-a}$
5. $L(\sin at) = \frac{a}{s^2+a^2}$
6. $L(\cos at) = \frac{s}{s^2+a^2}$
7. $L(\sinh at) = \frac{a}{s^-a^2}$
8. $L(\cosh at) = \frac{s}{s^2-a^2}$
9. $L[af(t) + bg(t)] = aL(f(t)) + bL(g(t))$
10. $L[af(t) - bg(t)] = aL(f(t)) - bL(g(t))$

Some basic Useful Results

1. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\,(a \sin bx - b \cos bx)$
2. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\,(a \cos bx + b \sin bx)$
3. $\sin^2 x = \frac{1-\cos 2x}{2}$
4. $\cos^2 x = \frac{1+\cos 2x}{2}$
5. $\sin^3 x = \frac{1}{4}\,[3 \sin x - \sin 3x]$
6. $\cos^3 x = \frac{1}{4}\,[\cos 3x - 3 \cos x]$
7. $\sin x \cos y = \frac{1}{2}\,[\sin(x+y) + \sin(x-y)]$
8. $\cos x \sin y = \frac{1}{2}\,[\sin(x+y) - \sin(x-y)]$
9. $\cos x \cos y = \frac{1}{2}\,[\cos(x+y) + \cos(x-y)]$
10. $\sin x \sin y = \frac{1}{2}\,[\cos(x-y) - \cos(x+y)]$

1.2 Problems Based On Transforms of Elementary Function

1. Find the Laplace transforms of following functions

   (a) $\sin 3t \cos 2t$

   (b) $\cos^3 2t$

   (c) $\sin 3t \sin 2t$

   (d) $sin^2 3t$

   (e) $e^{-4t} - 6t^2 + 4 \sin 2t$

   Ans.

   (a)

$$\begin{aligned}
L(\sin 3t \cos 2t) &= \frac{1}{2}\{L(\sin 5t) + L(\sin t)\} \\
&= \frac{1}{2}\{\frac{5}{s^2+25} + \frac{1}{s^2+1}\} \\
&= \frac{3(s^2+5)}{(s^2+1)(s^2+25)}
\end{aligned}$$

(b)

$$\begin{aligned} L(\sin^2 3t) &= L\left[\frac{1-\cos 6t}{2}\right] = \frac{1}{2}\{L(1) - L(\cos 6t)\} \\ &= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+36}\right\} = \frac{18}{s(s^2+36)} \end{aligned}$$

(c)

$$\begin{aligned} L(\sin 3t \sin 2t) &= \frac{1}{2}\{L(\cos t) - L(\cos 5t)\} \\ &= \frac{1}{2}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+25}\right\} = \frac{24}{(s^2+1)(s^2+25)} \end{aligned}$$

(d)

$$\begin{aligned} L(sin^2 3t) &= \frac{1}{2}\{1 - \cos 6t\} \\ &= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+36}\right\} = \frac{18}{s(s^2+36} \end{aligned}$$

(e)

$$\begin{aligned} L(e^{-4t} - 6t^2 + 4\sin 2t) &= L(e^{-4t} - 6L(t^2) + 4L(\sin 2t) \\ &= \frac{1}{s+4} - 6\frac{2!}{s^3} + 4\frac{2}{s^2+4} \\ &= \frac{1}{s+4} - \frac{12}{s^3} + \frac{8}{s^2+4} \end{aligned}$$

1.3 First Shifting Theorem

If $L(f(t)) = \phi(s)$, then $L(e^{at}f(t)) = \phi(s-a)$

1. Find the Laplace Transforms of the following

(a) $e^{-3t}t^3$

(b) $e^{-2t}\cos^2 t$

(c) $\sinh at \cdot \sin at$

(d) $e^{-2t}[\cos 4t + 3\sin 4t]$

(e) $(t+1)^2 e^t$

Ans.

(a)

$$\begin{aligned} We\ have\ L(t^3) &= \frac{3!}{s^4} \\ L(e^{-3t}t^3) &= \frac{6}{(s+3)^4}[\text{ replace s by s+3}] \end{aligned}$$

(b)

$$\begin{aligned} We\ have\ L(\cos 2t) &= \frac{s}{s^2+4} \\ L(e^{-2t}\cos^2 t) &= \frac{s+2}{(s+2)^2+4}[\text{ replace s by s+2}] \end{aligned}$$

(c) We have $\sinh at = \frac{e^{at}-e^{-at}}{2}$ and $L(\sin at) = \frac{a}{s^2+a^2}$

$$\begin{aligned} L(\sinh at \sin at) &= L(\frac{e^{at}-e^{-at}}{2}\sin at) \\ &= \frac{1}{2}\left\{e^{at}\sin at - e^{-at}\sin at\right\} - - - - - - - - (1) \\ L(e^{at}\sin at) &= \frac{a}{(s-a)^2+a^2} \qquad L(e^{-at}\sin at) = \frac{a}{(s+a)^2+a^2} \\ \text{form (1) } L(\sinh at \sin at) &= \frac{1}{2}\left\{\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right\} \end{aligned}$$

(d)

$$\begin{aligned} L(e^{-2t}[\cos 4t + 3\sin 4t]) &= L(e^{-2t}\cos 4t) + L(3e^{-2t}\sin 4t) \\ &= \frac{s+2}{(s+2)^2+16} + 3\frac{4}{(s+2)^2+16} \\ &= \frac{s+2}{(s+2)^2+16} + \frac{12}{(s+2)^2+16} = \frac{s+14}{s^2+4s+20} \end{aligned}$$

(e)

$$\begin{aligned} L((t+1)^2 e^t) &= L((t^2+2t+1)e^t) \\ &= L(e^t t^2) + 2L(e^t t) + L(e^t) \\ &= \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2} + \frac{1}{s-1} = \frac{s^2+1}{(s-1)^3} \end{aligned}$$