CONTENTS

LEARNING OBJECTIVES

After completing this lecture, students should be able to:

1. Understand the historical development and significance of Maxwell's contribution to electromagnetic theory.
2. Define electric potential and explain its role in electromagnetic fields.
3. State Maxwell's equations in their final integral and differential forms.
4. Interpret the physical meaning of each Maxwell equation in electric and magnetic fields.
5. Apply the constitutive relations for linear, homogeneous, and isotropic media to time-varying fields.
6. Explain the concept of continuity equation and its relation to Maxwell's equations.
7. Describe time-harmonic (sinusoidal) fields and their advantages in electromagnetic analysis.

8. Convert time-domain field expressions to phasor form and vice versa.
9. Use phasors to simplify Maxwell's equations for sinusoidal steady-state conditions.
10. Solve basic problems involving time-harmonic electric and magnetic fields using Maxwell's equations

1 INTRODUCTION

Electromagnetic theory forms the foundation of many modern engineering applications, including communication systems, medical instruments, and electronic devices. The most comprehensive and elegant formulation of this theory was developed by the Scottish physicist **James Clerk Maxwell (1831–1879)**, whose work unified electricity and magnetism into a single coherent framework.

Maxwell extended previous experimental results by introducing the concept of **displacement current**, which led to the theoretical prediction of **electromagnetic waves**. His four famous equations describe how electric and magnetic fields are generated and altered by charges, currents, and each other. Although initially met with skepticism, Maxwell's predictions were experimentally confirmed in 1888 by **Heinrich Hertz**, marking a major milestone in physics and engineering.

In this lecture, we explore Maxwell's equations in detail, beginning with electric potential and culminating in their **time-harmonic (phasor) forms**, which are essential for analyzing AC circuits, wave propagation, and electromagnetic behavior in real-world engineering systems.

2 MAXWELL'S EQUATIONS IN FINAL FORMS

The Scottish physicist James Clerk Maxwell (1831–1879) is regarded as the founder of electromagnetic theory in its present form. Maxwell's celebrated work led to the discovery of electromagnetic waves.4 Through his theoretical efforts when he was between 35 and 40 years old, Maxwell published the first unified theory of electricity and magnetism. The theory comprised all previously known results, both experimental and theoretical, on electricity and magnetism. It further introduced displacement current and predicted the existence of electromagnetic waves. Maxwell's equations were not fully accepted by many scientists until 1888, when they were confirmed by Heinrich Rudolf Hertz (1857–1894). The German physicist was successful in generating and detecting radio waves.

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampère's circuit law

Figure 1: Generalized Forms of Maxwell's Equations.

Is associated with Maxwell's equations. Also, the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \qquad (1)$$

Is implicit in Maxwell's equations. The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ, ε, and μ, the constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_o\mathbf{E} + \mathbf{P}$$
$$\mathbf{B} = \mu\mathbf{H} = \mu_o(\mathbf{H} + \mathbf{M}) \quad (3)$$
$$\mathbf{J} = \sigma\mathbf{E} + \rho_v\mathbf{u}$$

Hold for time-varying fields. Consequently, the boundary conditions remain valid for time-varying fields, where $\mathbf{a}_n$ is the unit normal vector to the boundary.

$$E_{1t} - E_{2t} = 0 \text{ or } (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_n = \mathbf{0}$$
$$H_{1t} - H_{2t} = K \text{ or } (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_n = \mathbf{K} \quad (3)$$
$$D_{1n} - D_{2n} = \rho_s \text{ or } (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_n = \rho_s$$
$$B_{1n} - B_{2n} = 0 \text{ or } (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_n = 0$$

3 TIME-HARMONIC FIELDS

A **time-harmonic field** is one that varies periodically or sinusoidally with time.

Not only is sinusoidal analysis of practical value, but also it can be extended to most waveforms by Fourier analysis. Sinusoids are easily expressed in phasors, which are more convenient to work with. Before applying phasors to EM fields, it is worthwhile to have a brief review of the concept of phasor:

A **phasor** is a complex number that contains the amplitude and the phase of a sinusoidal oscillation

As a complex number, a phasor z can be represented as:

$$z = x + jy = r\angle\phi$$
Or
$$z = re^{j\phi} = r(\cos\,\phi + j\sin\,\phi) \quad (2)$$

Where $j = \sqrt{-1}$, x is the real part of z, y is the imaginary part of z, r is the magnitude of z, and ϕ the phase of z, are given by:

$$r = |z| = \sqrt{x^2 + y^2}$$
$$\phi = tan^{-1}\frac{y}{x} \quad (3)$$

To introduce the time element, we let:

$$\phi = \omega t + \theta \quad (4)$$

Where θ may be a function of time or space coordinates or a constant.

$$re^{j\phi} = re^{j(\omega t+\theta)} = re^{j\theta}e^{j\omega t} \quad (5)$$

The real (Re) and imaginary (Im) parts are represented as following:

$$\mathrm{Re}\left(re^{j\phi}\right) = r\cos(\omega t + \theta)$$

and

$$\mathrm{Im}\left(re^{j\phi}\right) = r\sin(\omega t + \theta) \quad (6)$$

Thus, a sinusoidal current $I(t) = I_o\cos(\omega t + \theta)$, for example, equals the real part of $I_o e^{j\theta}e^{j\omega t}$. The current $I'(t) = I_o\sin(\omega t + \theta)$, which is the imaginary part of $I_o e^{j\theta}e^{j\omega t}$, can be represented as the real part of $I_o e^{j\theta}e^{j\omega t}e^{-j90^\circ}$ because $\sin\alpha = \cos(\alpha - 90^\circ)$.

The complex term $I_o e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the phasor current, denoted by I_s; that is,

$$I_s = I_o e^{j\theta} \quad (7)$$

Where the subscript s denotes the phasor form of $I(t)$. Thus $I(t) = I_o\cos(\omega t + \theta)$, the instantaneous form, can be expressed as:

$$I(t) = \mathrm{Re}\left(I_s e^{j\omega t}\right) \quad (8)$$

3.1 TIME DIFFERENTIATION AND INTEGRATION OF PHASOR FORM

In general, a phasor is a complex quantity and could be a scalar or a vector. If a vector $\mathbf{A}(x, y, z, t)$ is a time-harmonic field, the phasor form of $\mathbf{A}$ is $\mathbf{A}_s(x, y, z)$; the two quantities are related as:

$$\mathbf{A}(x,y,z,t) = \mathrm{Re}\left[\mathbf{A}_s(x,y,z)e^{j\omega t}\right] \qquad (9)$$

Note that the phasor is a function of position, not a function of time. For example, if $\mathbf{A} = A_0\cos(\omega t - \beta x)\mathbf{a}_y$, we can write $\mathbf{A}$ as:

$$\mathbf{A} = \mathrm{Re}\left(A_o e^{-j\beta x}\mathbf{a}_y e^{j\omega t}\right) \qquad (10)$$

The phasor form of $\mathbf{A}$ is:

$$\mathbf{A}_s = A_0 e^{-j\beta x}\mathbf{a}_y \qquad (11)$$

Time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$.

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial}{\partial t}\mathrm{Re}\left(\mathbf{A}_s e^{j\omega t}\right)\\ &= \mathrm{Re}\left(j\omega\mathbf{A}_s e^{j\omega t}\right)\end{aligned} \qquad (12)$$

That is in general:

$$\frac{\partial \mathbf{A}}{\partial t} \to j\omega\mathbf{A}_s$$
$$\int \mathbf{A}\partial t \to \frac{\mathbf{A}_s}{j\omega} \qquad (13)$$

3.2 TIME-HARMONIC MAXWELL'S EQUATIONS

We shall now apply the phasor concept to time-varying EM fields. The field quantities $\mathbf{E}(x,y,z,t), \mathbf{D}(x,y,z,t), \mathbf{H}(x,y,z,t), \mathbf{B}(x,y,z,t), \mathbf{J}(x,y,z,t)$, and $\boldsymbol{\rho}_v(x,y,z,t)$ and their derivatives can be expressed in phasor form:

Let,

$$\mathbf{E}(x,y,z,t) = \mathrm{Re}\left(\mathbf{E}_s(x,y,z)e^{j\omega t}\right)$$
$$\mathbf{B}(x,y,z,t) = \mathrm{Re}\left(\mathbf{B}_s(x,y,z)e^{j\omega t}\right) \tag{14}$$

Let us see how we can write Maxwell's equations in phasor form.

$$\nabla \times \mathbf{E}(x,y,z,t) = -\frac{\partial}{\partial t}\mathbf{B}(x,y,z,t) \tag{15}$$

Substituting (14) in (15) gives:

$$\nabla \times \left\{\mathrm{Re}\left[\mathbf{E}_s e^{j\omega t}\right]\right\} = -\frac{\partial}{\partial t}\left\{\mathrm{Re}\left[\ \mathbf{B}_s e^{j\omega t}\right]\right\} \tag{16}$$

We consider the left-hand side. The curl operation operates only on (x,y,z),

$$\nabla \times \left\{\mathrm{Re}\left[\mathbf{E}_s e^{j\omega t}\right]\right\} = \mathrm{Re}\left\{[\nabla \times \mathbf{E}_s]e^{j\omega t}\right\} \tag{17}$$

We similarly consider the right-hand side of (17). keeping in mind that $\mathbf{B}_s$ does not depend on time:

$$-\frac{\partial}{\partial t}\left\{\mathrm{Re}\left[\mathbf{B}_s e^{j\omega t}\right]\right\} = -\mathrm{Re}\left\{\mathbf{B}_s \frac{\partial}{\partial t} e^{j\omega t}\right\} = -\mathrm{Re}\left\{j\omega \mathbf{B}_s e^{j\omega t}\right\} \tag{18}$$

We obtain,

$$\nabla \times \mathbf{E}_s = -j\omega\ \mathbf{B}_s \tag{19}$$

Other Maxwell’s equations can be treated in a similar manner, and we obtain the following figure.

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

Figure 2: Time-Harmonic Maxwell's Equations Assuming Time Factor.

EXAMPLE 1 : Given that $\mathbf{A} = 10\cos(10^8 t - 10x + 60°)\mathbf{a}_z$ and $\mathbf{B}_s = (20/j)\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y$, express **A** in phasor form and $\mathbf{B}_s$ in instantaneous form.

Solution

$$\mathbf{A} = \text{Re}\left[10e^{j(\omega t - 10x + 60°)}\mathbf{a}_z\right]$$

where $\omega = 10^8$. Hence

$$\mathbf{A} = \text{Re}\left[10e^{j(60° - 10x)}\mathbf{a}_z e^{j\omega t}\right] = \text{Re}\left(\mathbf{A}_s e^{j\omega t}\right)$$

or

$$\mathbf{A}_s = 10e^{j(60° - 10x)}\mathbf{a}_z$$

If

$$\mathbf{B}_s = \frac{20}{j}\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y = -j20\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y$$
$$= 20e^{-j\pi/2}\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y$$
$$\mathbf{B} = \text{Re}\left(\mathbf{B}_s e^{j\omega t}\right)$$
$$= \text{Re}\left[20e^{j(\omega t - \pi/2)}\mathbf{a}_x + 10e^{j(\omega t + 2\pi x/3)}\mathbf{a}_y\right]$$
$$= 20\cos(\omega t - \pi/2)\mathbf{a}_x + 10\cos\left(\omega t + \frac{2\pi x}{3}\right)\mathbf{a}_y$$
$$= 20\sin\ \omega t\mathbf{a}_x + 10\cos\left(\omega t + \frac{2\pi x}{3}\right)\mathbf{a}_y$$

EXAMPLE 2 : In free space, Find **E** and β if

$$\boldsymbol{H} = 10\sin(10^8 t + \beta x)\mathbf{a}_y \text{ A/m}$$

Solution

Let $\omega = 10^8$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{0} + \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} \rightarrow \boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 40\sin(\omega t + \beta x) & 0 \end{vmatrix} = 40\beta\cos(\omega t + \beta x)\boldsymbol{a}_z$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt = \frac{40\beta}{\omega\varepsilon_o}\sin(\omega t + \beta x)\boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{40\beta}{\omega\varepsilon_0}\sin(\omega t + \beta x) \end{vmatrix} = -\frac{40\beta^2}{\omega\varepsilon_0}\cos(\omega t + \beta x)\boldsymbol{a}_y$$

$$-\mu_o\frac{\partial \boldsymbol{H}}{\partial t} = -40\mu_o\omega\cos(\omega t + \beta x)\boldsymbol{a}_y$$

then

$$40\mu_o\omega = \frac{40\beta^2}{\omega\varepsilon_o} \rightarrow \beta^2 = \mu_o\varepsilon_o\omega^2$$

$$\beta = \omega\sqrt{\varepsilon_o\mu_o} = 10^8\sqrt{4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi}} = \frac{1}{3} = \underline{\underline{0.333\text{rad/m}}}$$

EXAMPLE 3 : The electric field and the magnetic field in free space are given by

$$\mathbf{E} = \frac{50}{\rho}\cos(10^6 t + \beta z)\mathbf{a}_\phi \text{V/m}$$

$$\mathbf{H} = \frac{H_o}{\rho}\cos(10^6 t + \beta z)\mathbf{a}_\rho \text{A/m}$$

Express these in phasor form and determine the constants H_o and β such that the fields satisfy Maxwell's equations.

Solution

The instantaneous forms of **E** and **H** are written as

$$\mathbf{E} = \mathrm{Re}\left(\mathbf{E}_s e^{j\omega t}\right), \mathbf{H} = \mathrm{Re}\left(\mathbf{H}_s e^{j\omega t}\right)$$

where $\boldsymbol{\omega} = 10^6$ and phasors $\mathbf{E}_s$ and $\mathbf{H}_s$ are given by

$$\mathbf{E}_s = \frac{50}{\rho} e^{j\beta z} \mathbf{a}_\phi, \mathbf{H}_s = \frac{H_o}{\rho} e^{j\beta z} \mathbf{a}_\rho$$

For free space, $\rho_v = 0, \sigma = 0, \varepsilon = \varepsilon_0$, and $\mu = \mu_0$, so Maxwell's equations become

$$\nabla \cdot \mathbf{D} = \varepsilon_0 \nabla \cdot \mathbf{E} = 0 \ \rightarrow \nabla \cdot \mathbf{E}_s = 0$$
$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0 \rightarrow \nabla \cdot \mathbf{H}_s = 0$$
$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \rightarrow \nabla \times \mathbf{H}_s = j\omega\varepsilon_0 \mathbf{E}_s$$
$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \ \rightarrow \nabla \times \mathbf{E}_s = -j\omega\mu_0 \mathbf{H}_s$$
$$\nabla \cdot \mathrm{E}_s = \frac{1}{\rho} \frac{\partial}{\partial \phi}\left(E_{\phi s}\right) = 0$$
$$\nabla \cdot \mathrm{H}_s = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho H_{\rho s}\right) = 0$$

Now

$$\nabla \times \mathbf{H}_s = \nabla \times \left(\frac{H_o}{\rho} e^{j\beta z} \mathbf{a}_\rho\right) = \frac{jH_o\beta}{\rho} e^{j\beta z} \mathbf{a}_\phi$$

$$\frac{jH_o\beta}{\rho} e^{j\beta z} \mathbf{a}_\phi = j\omega\varepsilon_o \frac{50}{\rho} e^{j\beta z} \mathbf{a}_\phi$$

or

$$H_o\beta = 50\omega\varepsilon_o$$

Similarly,

$$-j\beta \frac{50}{\rho} e^{j\beta z} \mathbf{a}_\rho = -j\omega\mu_o \frac{H_o}{\rho} e^{j\beta z} \mathbf{a}_\rho$$

or

$$\frac{H_o}{\beta} = \frac{50}{\omega\mu_o}$$

$$H_o^2 = (50)^2 \frac{\varepsilon_o}{\mu_o}$$

or

$$H_o = \pm 50\sqrt{\varepsilon_o/\mu_o} = \pm \frac{50}{120\pi} = \pm 0.1326$$

we get

$$\beta^2 = \omega^2 \mu_o \varepsilon_o$$

or

$$\beta = \pm\omega\sqrt{\mu_0\varepsilon_0} = \pm\frac{\omega}{c} = \pm\frac{10^6}{3\times 10^8}$$
$$= \pm 3.33\times 10^{-3}$$

$H_0 = 0.1326, \beta = 3.33\times 10^{-3}$

or $H_0 = -0.1326, \beta = -3.33\times 10^{-3}$;

REFERENCES

- Ida, Nathan. Engineering Electromagnetics. 3rd edition, Springer International Publishing, 2015.
- Sadiku, Matthew N. O. *Elements of Electromagnetics*. Seventh edition, Oxford University Press, 2018. Oxford Series in Electrical and Computer Engineering.
- Hayt, William Hart, and John A. Buck. Engineering Electromagnetics. Ninth edition, McGraw-Hill Education, 2019.