CONTENTS

LEARNING OBJECTIVES

By the end of this lecture, students will be able to:

1. Explain the fundamental concepts of **electrostatics**.
2. Define **electric charge** and describe its types, properties, and conservation law.
3. Apply **Coulomb's Law** to calculate the magnitude and direction of electrostatic forces between point charges.
4. Understand and use the principle of **superposition** to determine the resultant electric force or field due to multiple charges.
5. Define and compute **electric field intensity** generated by discrete and continuous charge distributions.
6. Identify and analyze different types of charge distributions (**line, surface, and volume**).

1 INTRODUCTION

Electrostatics is the branch of electromagnetics concerned with **stationary or slow-moving electric charges**. It forms the foundation of understanding how electric fields interact with materials and is essential for grasping advanced electromagnetic concepts. In electrostatics, we study how charges produce electric fields and how these fields exert forces on other charges.

The behavior of static charges explains many everyday phenomena, such as the attraction of a charged comb to small paper pieces or the spark from touching a metal object after walking on a carpet. In engineering—especially **medical instrumentation**—electrostatics plays a critical role in designing devices such as **electrocardiographs (ECG), electrophoresis systems, and imaging equipment** that depend on controlled electric fields.

Through this lecture, students will build a foundational understanding of how charges interact, how electric fields are generated, and how to mathematically describe these relationships using **Coulomb's law**, **field intensity**, and **charge distribution analysis**

2 ELECTROSTATIC

Electrostatics is a branch of physics that studies slow-moving or stationary electric charges on macroscopic objects where quantum effects can be neglected.

Under these circumstances the electric field, electric potential, and the charge density are related without complications from magnetic effects.

Examples of Electrostatic Phenomena are as follows:

- A balloon rubbing hair.
- The shock of touching a doorknob after crossing a carpet.
- An electric balloon adhering to a wall.

- A charged comb that gathers tiny bits of paper.
- rubbing nylon clothing against flesh or other materials.
- Using a towel to rub a rod.
- Utilizing a TV screen.
- Putting on winter clothing.
- Making use of a photocopier.

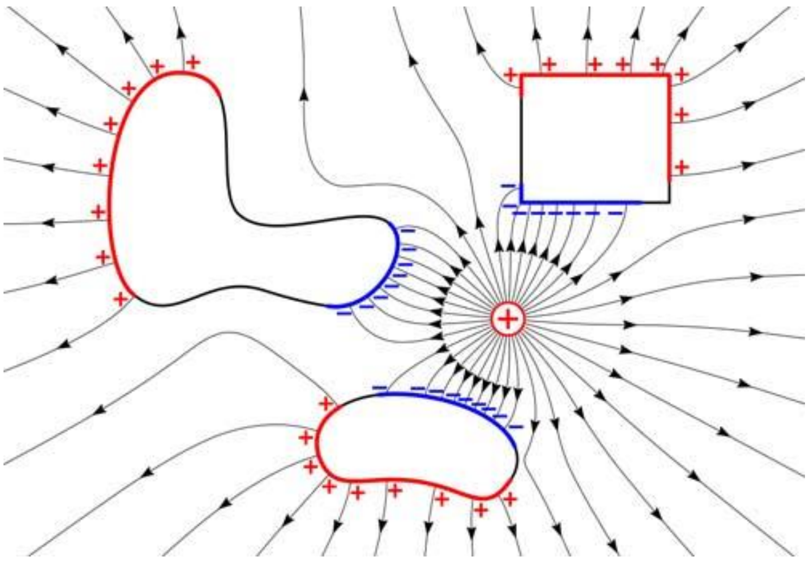


Figure 1: Electric Field Lines due to different charges.

3 ELECTRIC CHARGE

Electric charge is a fundamental property of matter that determines how it interacts with electromagnetic fields. When charges are stationary, they produce an electric field around them, and when in motion, they produce a magnetic field as well. Electric charge comes in two types: positive and negative. Like charges repel whereas unlike charges attract.

Charge is a scalar and is measured in coulombs which is the unit of the charge in the International System of Units (SI) of charge with the symbol of (C).

One coulomb is the electric charge delivered by a 1 ampere current in 1 second.

3.1 TYPES OF ELECTRIC CHARGE

Electric charge comes in two main types: **positive** and **negative** charges. Positive charges are associated with protons, which are subatomic particles residing in the nucleus of an atom. They are represented by the symbol "+". On the other hand, negative charges are linked to electrons, which orbit the atomic nucleus and are denoted by the symbol "-".

3.2 PROPERTIES OF ELECTRIC CHARGE

- **Charge is quantized**. This means that electric charge comes in discrete amounts, and there is a smallest possible amount of charge that an object can have. That is:

$$Q = ne \qquad (1)$$

Where $n = \pm 1, \pm 2, \pm 3 \ldots\ldots$ and $\boldsymbol{e = 1.6 \times 10^{-19} C}$ is the fundamental unit of charge (elementary charge).

No free particle can have less charge than this, and, therefore, the charge on any object—the charge on all objects—must be an integer multiple of this amount.

- **Additivity** - The net electric charge of a system is equal to an algebraic sum of individual electric charges present in the system.
- **Conservation of electric charge** - Net electric charge of an isolated system remains conserved.

3.3 COULOMB'S LAW

Coulomb's law is an experimental law formulated in 1785 by Charles Augustin de Coulomb, then a colonel in the French army. It deals with the force a point charge exerts on another point charge.

By a *point charge* we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions. For example, a collection of electric charges on a pinhead may be regarded as a point charge. Electrons are regarded as point charges. The polarity of charges may be positive or negative; like charges repel, while unlike charges attract.

Coulomb's Law says that "The magnitude of the electrostatic force between two-point charges (either attraction or repulsion) is directly proportional to the product of the magnitudes of charges and is inversely proportional to the square of the distance between them."

Mathematically,

$$F = \frac{kQ_1Q_2}{R^2} \qquad (2)$$

Where Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and k is a proportionality constant. If the International System of Units (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as:

$$k = \frac{1}{4\pi\varepsilon_0} \qquad (3)$$

The new constant ε_0 is called the *permittivity of free space* and has magnitude, measured in farads per meter (F/m),

$$\varepsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi}10^{-9} \text{ F/m} \qquad (4)$$

The quantity ε_0 is not dimensionless, for Coulomb's law shows that it has the label $(C^2/N.m^2)$. Then we can define Coulomb's law as follows:

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \tag{5}$$

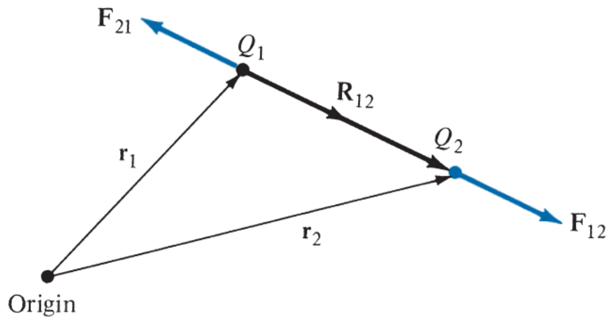


Figure 2: Coulomb vector force on point charges Q_1 and Q_2.

If point charges Q_1 and Q_2 are located at points having position vectors r_1 and r_2, then the vector form of coulomb's law of the force F_{12} on Q_2 due to Q_1, shown in Figure 2, is given by:

$$\boldsymbol{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\boldsymbol{a}_{R_{12}} \tag{6}$$

Where;

$$\boldsymbol{R}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \quad R = |\boldsymbol{R}_{12}| \quad , \quad \boldsymbol{a}_{R_{12}} = \frac{\boldsymbol{R}_{12}}{R} \tag{7}$$

By substituting in the vector form equation, we may write Equation (6) as:

$$\boldsymbol{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\frac{\boldsymbol{R}_{12}}{R}$$
$$= \frac{Q_1 Q_2}{4\pi\varepsilon_0}\frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} \tag{8}$$

EXAMPLE 1 : If a charge of $Q_1 = 3 \times 10^{-4}$C is located at $M(1,2,3)$ and a charge of $Q_2 = -10^{-4}$C at $N(2,0,5)$ in a vacuum. Find the force exerted on Q_2 by Q_1?

Solution

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{N}$$

Issues concerning Coulomb's Law:

1) As shown in Figure 2, the force $\mathbf{F}_{12}$ on Q_1 due to Q_2 is given by

$$\boldsymbol{F}_{21} = |\boldsymbol{F}_{12}|\boldsymbol{a}_{R_{21}} = |\boldsymbol{F}_{12}|(-\boldsymbol{a}_{R_{12}}) = -\boldsymbol{F}_{12} \qquad (9)$$

2) Like charges (charges of the same sign) repel each other, while unlike charges attract.

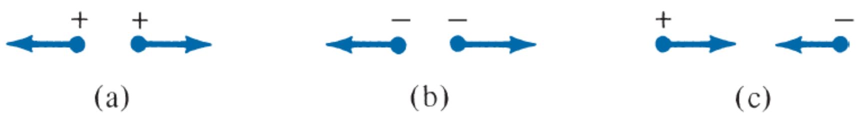


Figure 3: (a), (b) Like charges repel. (c) Unlike charges attract.

3) The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charges.

4) Q_1 and Q_2 must be static (at rest).

5) The signs of Q_1 and Q_2 must be taken into account in Equation (6) For like charges, $Q_1Q_2 > 0$. For unlike charges, $Q_1Q_2 < 0$.

6) Charges cannot be created or destroyed; the quantity of total charge remains constant.

EXAMPLE 2 : Direction of Forces Between Charges Two charges, Q_1 and Q_2, are located at points $P_1(1,1,0)$ and $P_2(3,2,0)$:

(a) Calculate the force on Q_1 and Q_2 if $Q_1 = 2 \times 10^{-9}$C and $Q_2 = 4 \times 10^{-9}$C.

(b) Calculate the force on Q_1 and Q_2 if $Q_1 = 2 \times 10^{-9}$C and $Q_2 = -4 \times 10^{-9}$C

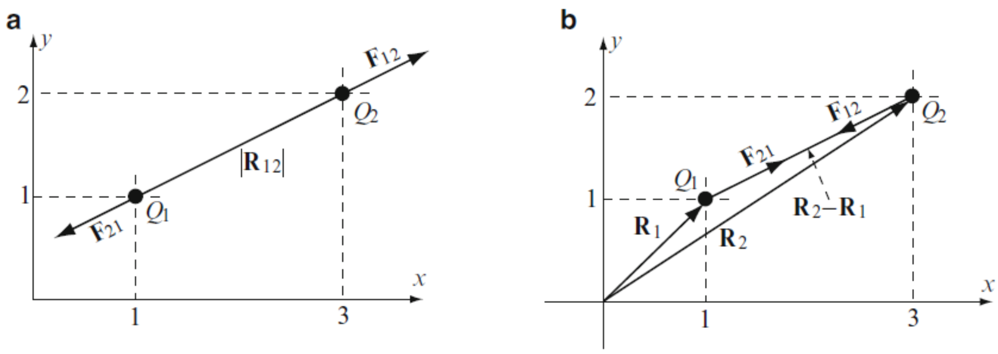


Figure 4: (a) Forces between two positive or two negative charges. (b) Forces between a positive and a negative charge.

Solution

(a)

$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\mathbf{R}_{12}, \qquad \mathbf{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\mathbf{R}_{21}$$

Calculation of the unit vector $\mathbf{R}_{12}$ and $\mathbf{R}_{21}$:

$$\mathbf{R}_{12} = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y = (3-1)\mathbf{a}_x + (2-1)\mathbf{a}_y = 2\mathbf{a}_x + \mathbf{a}_y$$

$$\mathbf{R}_{21} = -\mathbf{R}_{12} = -2\mathbf{a}_x - \mathbf{a}_y$$

The unit vectors are

$$\mathbf{a}_{R_{12}} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{2\mathbf{a}_x - \mathbf{a}_y 1}{\sqrt{5}}, \qquad \mathbf{a}_{R_{21}} = \frac{-2\mathbf{a}_x - \mathbf{a}_y}{\sqrt{5}}$$

Thus, the force on Q_1 (i.e., the force $\mathbf{F}_{21}$, exerted by charge Q_2 on Q_1) is

$$\mathbf{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 \mathrm{R}_{21}^2}\mathbf{a}_{R_{21}} = \left(\frac{2 \times 10^{-9} \times 4 \times 10^{-9}}{4 \times \pi \times 8.854 \times 10^{-12} \times 5}\right)\frac{-2\mathbf{a}_x - \mathbf{a}_y}{\sqrt{5}}$$

$$= 12.86 \times 10^{-9}\mathbf{a}_x - 6.43 \times 10^{-9}\mathbf{a}_y$$

From the fact that $\mathbf{F}_{12} = -\mathbf{F}_{21}$, we get the force on Q_2 as

$$\mathbf{F}_{12} = 12.86 \times 10^{-9}\mathbf{a}_x + 6.43 \times 10^{-9}\mathbf{a}_y \quad \mathrm{N}$$

(b)

$$\mathbf{R}_1 = (x_1 - 0)\mathbf{a}_x + (y_1 - 0)\mathbf{a}_y = \mathbf{a}_x + \mathbf{a}_y$$
$$\mathbf{R}_2 = (x_2 - 0)\mathbf{a}_x + (y_2 - 0)\mathbf{a}_y = 3\mathbf{a}_x + 2\mathbf{a}_y$$

$$\mathbf{R}_{12} = \mathbf{R}_1 - \mathbf{R}_2 = 2\mathbf{a}_x + \mathbf{a}_y$$

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (2\mathbf{a}_x + \mathbf{a}_y)}{4\pi\varepsilon_0 |2\mathbf{a}_x + \mathbf{a}_y|^3} = \left(\frac{2\times 10^{-9} \times (-4\times 10^{-9})}{4\times\pi\times 8.854\times 10^{-12}}\right)\frac{2\mathbf{a}_x + \mathbf{a}_y}{5\sqrt{5}}$$
$$= -12.86\times 10^{-9}\mathbf{a}_x - 6.43\times 10^{-9}\,\mathbf{a}_y \quad \text{N}$$

and

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = \hat{\mathbf{x}}12.86\times 10^{-9} + \hat{\mathbf{y}}6.43\times 10^{-9} \qquad \text{N}$$

3.4 PRINCIPLE OF SUPERPOSITION

The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_N}$, the resultant force F on a charge Q located at point $\mathbf{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$.

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots + \boldsymbol{F}_N \\ &= \frac{QQ_1(\boldsymbol{r}-\boldsymbol{r}_1)}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{r}_1|^3} + \frac{QQ_2(\boldsymbol{r}-\boldsymbol{r}_2)}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{r}_2|^3} + \cdots + \frac{QQ_N(\boldsymbol{r}-\boldsymbol{r}_N)}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{r}_N|^3} \\ &= \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\boldsymbol{r}-\boldsymbol{r}_k)}{|\boldsymbol{r}-\boldsymbol{r}_k|^3} \end{aligned} \qquad (10)$$

EXAMPLE 3 : Point charges 1 mC and -2 mC are located at (3,2, -1) and (-1, -1,4), respectively. Calculate the electric force on a 10 nC charge located at (0,3,1)?

Solution

$$\mathbf{F} = \sum_{k=1,2}\frac{QQ_k}{4\pi\varepsilon_0 R^2}\mathbf{a}_R = \sum_{k=1,2}\frac{QQ_k(\mathbf{r}-\mathbf{r}_k)}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_k|^3}$$
$$= \frac{Q}{4\pi\varepsilon_0}\left\{\frac{10^{-3}[(0,3,1)-(3,2,-1)]}{|(0,3,1)-(3,2,-1)|^3} - \frac{2\cdot 10^{-3}[(0,3,1)-(-1,-1,4)]}{|(0,3,1)-(-1,-1,4)|^3}\right\}$$
$$= \frac{10^{-3}\cdot 10\cdot 10^{-9}}{4\pi\cdot\frac{10^{-9}}{36\pi}}\left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}}\right]$$
$$= 9\cdot 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right]$$

$$\mathbf{F} = -6.512\mathbf{a}_x - 3.713\mathbf{a}_y + 7.509\mathbf{a}_z m\text{N}$$

4 ELECTRICAL FIELD INTENSITY

The *electric field intensity* gives the magnitude and direction of electrostatic force that would be applied to a point charge of unit magnitude that resides in the field, and as a function of its location. Emphasized here is the notion of the force acting *at a point,* and as such, the electric field intensity, like all other field quantities we will encounter, is a *point function.*

We can define *electric field intensity as:*

The **electric field intensity** (or electric field strength) E is the force that a unit positive charge experiences when placed in an electric field.

Thus;

$$\mathbf{E} = \frac{\mathbf{F}}{Q} \tag{11}$$

For $Q > 0$, the electric field intensity $\mathbf{E}$ is obviously in the direction of the force $\mathbf{F}$ and is measured in newtons per coulomb or volts per meter. The electric field intensity at point $\mathbf{r}$ due to a point charge located at $\mathbf{r}'$ is readily obtained as:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R, \qquad \text{or } \mathbf{E} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|^3} \tag{12}$$

For N point charges $Q_1, Q_2, \dots, Q_N$ located at $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$, the electric field intensity at point $\mathbf{r}$ is obtained as:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots + \mathbf{E}_N$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r}-\mathbf{r}_k)}{|\mathbf{r}-\mathbf{r}_k|^3} \tag{13}$$

The units of **E** would be in force per unit charge newtons per coulomb (N/C) or volts per meter (V/m).

EXAMPLE 4 : Find **E** at $P(1,1,1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1,1,0), P_2(-1,1,0), P_3(-1,-1,0)$, and $P_4(1,-1,0)$, as shown in Figure 5?

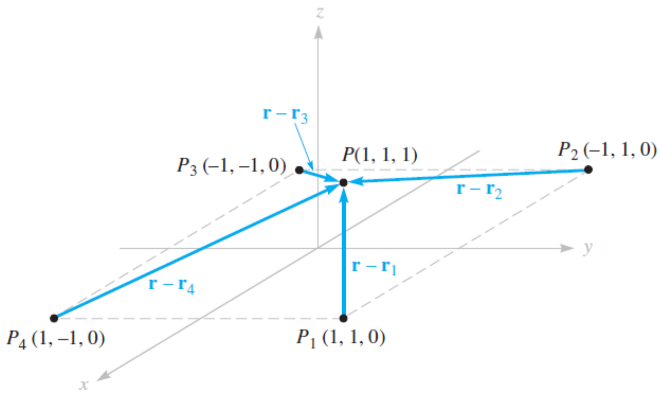


Figure 5: A symmetrical distribution of four identical 3-nC point.

Solution

$\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z, \mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$.

The magnitudes are:

$|\mathbf{r} - \mathbf{r}_1| = 1, |\mathbf{r} - \mathbf{r}_2| = \sqrt{5}, |\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$.

Because $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\ \text{V} \cdot \text{m}$,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}$$

$$\mathbf{E} = 26.96 \left[\frac{\mathbf{a}_z}{1} \frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3} \frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} \right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z\ \text{V/m}$$

4.1 ELECTRIC FIELD LINES

In an attempt to visualize the electric field, it is customary to draw the electric field intensity in terms of field lines. These are imaginary lines that show the direction of force on an infinitesimal positive point charge if it were placed in the field. The electric field intensity is everywhere tangential to field lines. Field lines can also be called force lines. Plots of field lines are quite useful in describing, qualitatively, the behavior of the electric field and of charges in the electric field.

The electric field intensity of the point charge in Figure 6 shows the electric field lines for a positive and a negative point charge. Similar sketches of more complicated field distributions help in understanding the field distribution in space. For example, Figure 7 shows the field lines of two equal but opposite point charges. The following should be noted from this description of the field:

- Field lines begin at positive charges and end in negative charges. If only one type of charge exists, the lines start or end at infinity .
- Field lines show the direction of force on a positive point charge if it were placed in the field and, therefore, also show the direction of the electric field intensity. The arrows help in showing the direction of force and field.
- Field lines are imaginary lines; their only purpose is to visualize the electric field

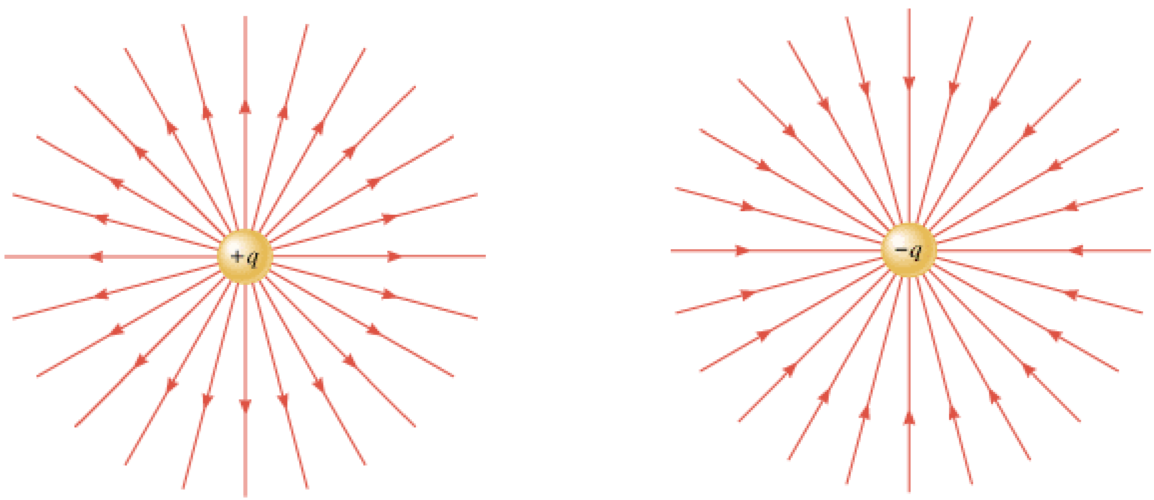


Figure 6: Field lines of point charges.

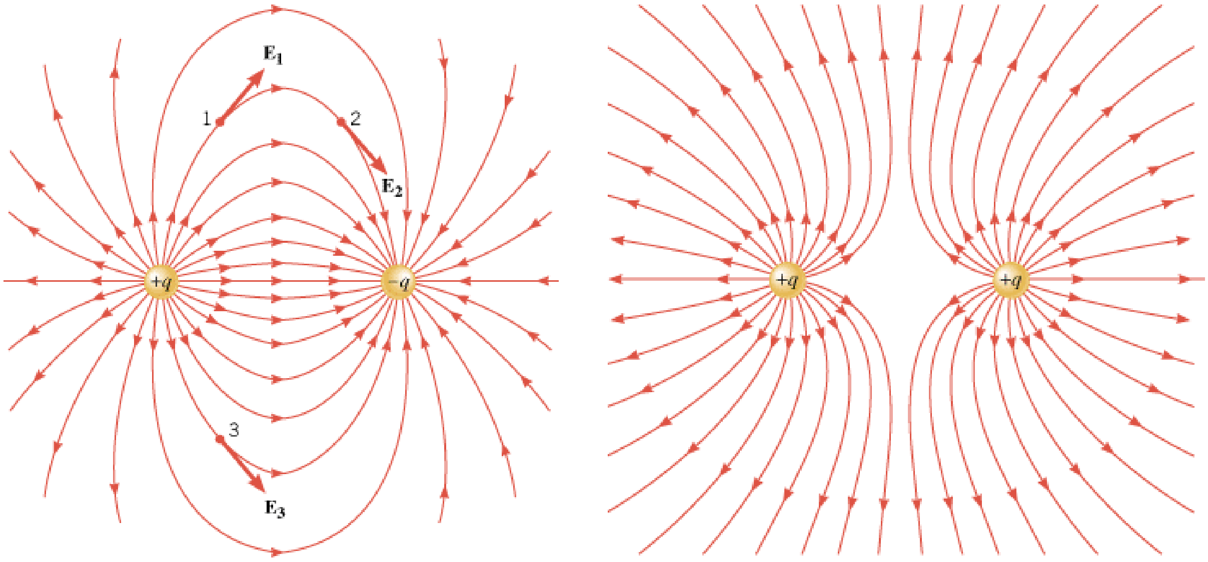


Figure 7: Field lines between two-point charges.

5 ELECTRIC FIELDS DUE TO CONTINUOUS CHARGE DISTRIBUTIONS

So far, we have considered only forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume, as illustrated below.

It is customary to denote the line charge density, surface charge density, and volume charge density by ρ_L (in $\mathrm{C/m}$), ρ_S (in $\mathrm{C/m^2}$), and ρ_v (in $\mathrm{C/m^3}$), respectively. These must not be confused with ρ (without subscript), used for radial distance in cylindrical coordinates.

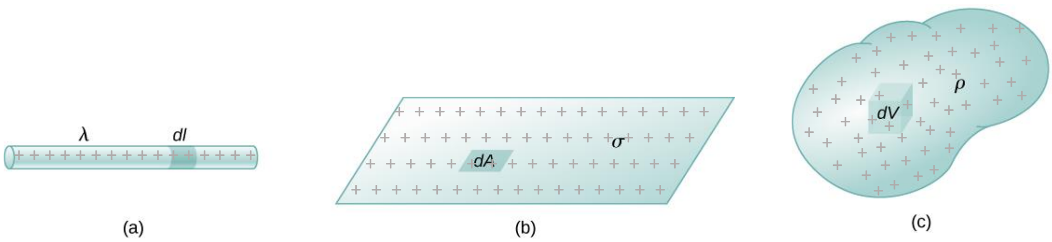


Figure 8: Various charge distributions (a) Line (b) Surface (c) Volume.

The charge element dQ and the total charge Q due to these charge distributions are obtained as:

$$dQ = \rho_L dl \quad \rightarrow \quad Q = \int_L \rho_L dl \ \ \text{(line charge)}$$
$$dQ = \rho_S dS \quad \rightarrow \quad Q = \int_S \rho_S dS \ \ \text{(surface charge)} \qquad (14)$$
$$dQ = \rho_v dv \quad \rightarrow \quad Q = \int_v \rho_v dv \ \ \text{(volume charge)}$$

The electric field intensity due to each of the charge distributions:

$$\boldsymbol{E} = \int_L \frac{\rho_L dl}{4\pi\varepsilon_o R^2} \boldsymbol{a}_R \ \ \text{(line charge)}$$
$$\boldsymbol{E} = \int_S \frac{\rho_S dS}{4\pi\varepsilon_o R^2} \boldsymbol{a}_R \ \ \text{(surface charge)} \qquad (15)$$
$$\boldsymbol{E} = \int_v \frac{\rho_v dv}{4\pi\varepsilon_o R^2} \boldsymbol{a}_R \ \ \text{(volume charge)}$$

EXAMPLE 5 : Find the total charge contained in a 2-cm length of the electron beam shown below?

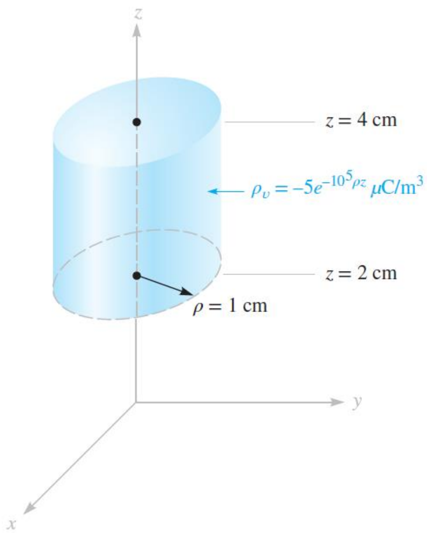


Figure 9: The total charge contained within the right circular cylinder.

Solution

We see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \mathrm{C/m^2}$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04}\int_{0}^{2\pi}\int_{0}^{0.01} -5\times 10^{-6} e^{-10^5\rho z}\rho d\rho d\phi dz$$

We integrate first with respect to ϕ because it is so easy,

$$Q = \int_{0.02}^{0.04}\int_{0}^{0.01} -10^{-5}\pi e^{-10^5\rho z}\rho d\rho dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_{0}^{0.01}\left(\frac{-10^{-5}\pi}{-10^5\rho}e^{-10^5\rho z}\rho d\rho\right)_{z=0.02}^{z=0.04}$$

$$= \int_{0}^{0.01} -10^{-5}\pi(e^{-2000\rho} - e^{-4000\rho})d\rho$$

Finally,

$$Q = -10^{-10}\pi\left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000}\right)_0^{0.01}$$

$$Q = -10^{-10}\pi\left(\frac{1}{2000} - \frac{1}{4000}\right) = \frac{-\pi}{40} = 0.0785\text{pC}$$

where pC indicates picocoulombs.

EXAMPLE 6 : A circular ring of radius a carries a uniform charge ρ_LC/m and is placed on the xy-plane with axis the same as the z-axis.

(a) Show that

$$\text{E}(0,0,h) = \frac{\rho_L a h}{2\varepsilon_o[h^2 + a^2]^{3/2}}\mathbf{a}_z$$

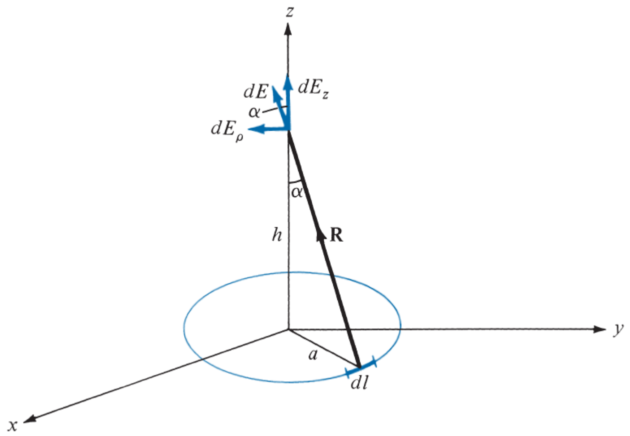
(b) If the total charge on the ring is Q, find E as $a \to 0$.

Solution

(a)

$$dl = ad\phi, \quad \mathbf{R} = a(-\mathbf{a}_\rho) + h\mathbf{a}_z$$

$$R = |\mathbf{R}| = [a^2 + h^2]^{\frac{1}{2}}, \qquad \mathbf{a}_R = \frac{\mathbf{R}}{R}$$



$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{-a\mathbf{a}_\rho + h\mathbf{a}_z}{[a^2 + h^2]^{3/2}}$$

Hence

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi} \frac{\left(-a\mathbf{a}_\rho + h\mathbf{a}_z\right)}{[a^2 + h^2]^{3/2}} a d\phi$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero. This is evident from the fact that for every element dl there is a corresponding element diametrically opposite that gives an equal but opposite dE_ρ so that the two contributions cancel each other. Thus, we are left with the z-component. That is,

$$\mathbf{E} = \frac{\rho_L a h \mathbf{a}_z}{4\pi\varepsilon_o[h^2 + a^2]^{3/2}}\int_0^{2\pi} d\phi = \frac{\rho_L a h \mathbf{a}_z}{2\varepsilon_o[h^2 + a^2]^{3/2}}$$

(b) Since the charge is uniformly distributed, the line charge density is:

$$\rho_L = \frac{Q}{2\pi a}$$

so that

$$\mathbf{E} = \frac{Qh}{4\pi\varepsilon_o[h^2 + a^2]^{3/2}}\mathbf{a}_z$$

As $a \to 0$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o h^2}\mathbf{a}_z$$

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