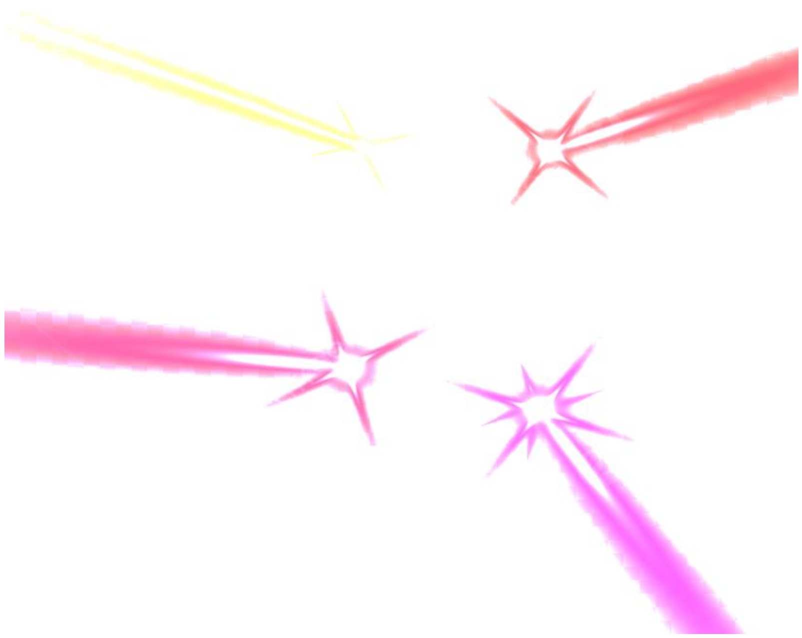
Lecture 2
Einstein Coefficients and the Principle of Laser Amplification



Lecturer:
MSc. Huda Wasfi Hassoon

Email: (huda.wasfi@uomus.edu.iq)

When light interacts with matter, atoms can absorb or emit photons depending on their internal energy levels. Albert Einstein (1917) introduced a theoretical model describing these processes using three parameters: **A**, **B_{12}**, and **B_{21}**, known as the **Einstein coefficients**.

These coefficients explain the mechanisms of absorption, spontaneous emission, and stimulated emission, which together form the physical basis of laser operation.

Atomic Energy Levels and Photon Interaction

Consider an atom with two discrete energy levels:

- Lower energy level: E_1
- Upper energy level: E_2

The energy difference between these levels corresponds to a photon of frequency ν:

$$h\nu = E_2 - E_1$$

where:

- h= Planck's constant (6.626×10^{-34} J.s)
- ν= frequency of radiation (Hz)

Atoms interact with radiation through three fundamental processes, as shown below.

(a) Absorption

An atom in the lower energy level E_1absorbs a photon of energy $h\nu$ and is excited to the upper level E_2.

The rate of absorption per unit volume is:

Email: (huda.wasfi@uomus.edu.iq)

$$R_{\text{abs}} = N_1 \rho(\nu) B_{12}$$

where:

- N_1: population density of atoms in E_1
- $\rho(\nu)$: radiation energy density at frequency ν ($J \cdot m^{-3} \cdot Hz^{-1}$)
- B_{12}: Einstein coefficient for **absorption** ($m^3 \cdot J^{-1} \cdot s^{-2}$)

(b) Spontaneous Emission

An atom in the excited level E_2 may spontaneously return to the lower level E_1, emitting a photon with energy $h\nu$.

The rate of spontaneous emission per unit volume is:

$$R_{\text{spont}} = N_2 A_{21}$$

where:

- N_2: population density of atoms in E_2
- A_{21}: Einstein coefficient for **spontaneous emission** (s^{-1})

The emitted photons are random in direction and phase, leading to incoherent light (as in ordinary lamps).

(c) Stimulated Emission

When a photon of energy $h\nu$ interacts with an excited atom (E_2), it can stimulate the atom to emit a second photon that is **identical** to the first (same direction, phase, and frequency).

The rate of stimulated emission per unit volume is:

$$R_{\text{stim}} = N_2 \rho(\nu) B_{21}$$

where:

- B_{21}: Einstein coefficient for **stimulated emission**
- N_2: population density of atoms in E_2
- $\rho(\nu)$: radiation energy density at frequency ν ($J \cdot m^{-3} \cdot Hz^{-1}$)

This process is responsible for **light amplification** in lasers.

Einstein's Relations in Thermal Equilibrium

1- The Fundamental Balance

In thermal equilibrium, the population densities N_1 and N_2 (the number of atoms per unit volume in the lower and upper energy levels, respectively) remain constant over time. For this steady state to be maintained, the rate at which atoms move upwards must equal the rate at which they move downwards.

The Balance Equation:

Rate of Absorption = Rate of Stimulated Emission + Rate of Spontaneous Emission

$$N_1 \rho(\nu) B_{12} = N_2 \rho(\nu) B_{21} + N_2 A_{21} \qquad \text{(Equation 1)}$$

2. How Atoms are Distributed: The Boltzmann Distribution

It is a fundamental law of statistical mechanics that tells us how particles are spread out among different energy levels when a system is in thermal equilibrium at

Email: (huda.wasfi@uomus.edu.iq)

a given temperature T. Its core principle is that lower energy levels are always more populated than higher ones.

The general form of the **Boltzmann Distribution** is:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT}$$

What is Degeneracy? Degeneracy (often called the "statistical weight") is the number of distinct quantum states that an atom can have at the same energy level.

For simplicity, if we assume the degeneracies are equal ($g_1 = g_2$), meaning both levels have the same number of "quantum seats," the distribution simplifies to:

$$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/kT}$$

Which can be rearranged as:

$$\frac{N_1}{N_2} = e^{(E_2 - E_1)/kT} = e^{h\nu/kT} \qquad \text{(Equation 2)}$$

This shows clearly that the population of the upper level (N_2) decreases exponentially as the energy gap ($h\nu$) increases or the temperature (T) decreases.

Where:

k: Boltzmann constant ($1.38 \times 10^{-23} J/K$)

3. Deriving Einstein's Relations

We can solve Equation 1 for the energy density $\rho(\nu)$:

$$\rho(\nu) = \frac{A_{21}/B_{21}}{(B_{12}/B_{21})(N_1/N_2) - 1}$$

Now, we substitute the population ratio from the **Boltzmann Distribution** (Equation 2), which gives us the ratio of atoms in the lower state to the upper state based on the temperature and energy difference:

$$\rho(\nu) = \frac{A_{21}/B_{21}}{(B_{12}/B_{21})e^{h\nu/kT} - 1} \qquad \text{(Equation 3)}$$

This is our derived formula for the energy density of radiation in equilibrium with atoms.

4. Comparison with Planck's Law: Planck's Blackbody Radiation

It is the thermal radiation emitted by an idealized, perfect absorber and emitter called a **blackbody**. A key feature is that the spectrum and intensity of this radiation depend *only* on the body's temperature, not its material. Planck's law is the quantum mechanical formula that correctly describes this spectrum.

Planck's law for the energy density of **blackbody radiation** is:

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/kT} - 1} \qquad \text{(Equation 4)}$$

5. The Final Step: Equating the Two Forms

For our derived Equation 3 to be physically valid and match the established **Planck's Blackbody Radiation** law (Equation 4) for all temperatures T and frequencies ν, the coefficients in front must be identical. This requirement leads to Einstein's famous relations:

1. $B_{12} = B_{21}$
   - **Physical Meaning:** The probability per unit atom for absorption and stimulated emission is equal. This is a profound result.
2. $\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}$
   - **Physical Meaning:** The rate of spontaneous emission is proportional to the cube of the frequency. This explains why high-energy transitions (e.g., in X-ray lasers) have very short lifetimes and are dominated by spontaneous emission.

6. Ratio of Spontaneous to Stimulated Emission

The ratio R determines which process—random spontaneous emission or ordered stimulated emission—dominates the system's behavior.

$$R = \frac{\text{Spontaneous Emission Rate}}{\text{Stimulated Emission Rate}} = \frac{N_2 A_{21}}{N_2 \rho(\nu) B_{21}} = \frac{A_{21}}{\rho(\nu) B_{21}}$$

Substituting the relation $A_{21}/B_{21} = 8\pi h\nu^3/c^3$ and the blackbody formula for $\rho(\nu)$ from Equation 4, we find the beautifully simple result:

Email: (huda.wasfi@uomus.edu.iq)

$$R = e^{h\nu/kT} - 1$$

Population Inversion and Optical Gain

To achieve light amplification, we must create a **population inversion**:

$$N_2 > N_1$$

This condition is *non-equilibrium* and is achieved using an **external pumping source** (optical, electrical, or chemical).

When a light beam passes through such a medium, the change in its intensity $I(x)$along a small distance $dx$is given by:

$$\frac{dI}{dx} = -\alpha I$$

where:

- $I(x)$: **Intensity** of the light beam at position x.
- $\frac{dI}{dx}$: **Rate of change** of intensity with distance.
- α: **Absorption coefficient** (measured in m^{-1}), which is proportional to $N_1 - N_2$.

Thus:

- If $N_1 > N_2$, $\alpha > 0$→ absorption dominates.
- If $N_2 > N_1$, $\alpha < 0$→ gain occurs.

Define the **gain coefficient** $g = -\alpha > 0$, then:

Email: (huda.wasfi@uomus.edu.iq)

$$I(x) = I_0 e^{gx}$$

- I_0: Initial intensity of the light at $x = 0$.
- $I(x)$: Intensity after traveling distance x through the gain medium.
- e^{gx}: **Exponential growth factor** that describes how much the light is amplified.

This exponential growth describes **optical amplification** — the fundamental mechanism in laser operation.

Example: He–Ne Laser Transition

Consider a helium-neon (He-Ne) laser operating at the red transition line with a frequency of $\nu = 4.74 \times 10^{14}$ Hz. The laser tube is maintained at a temperature of $T = 370$ K.

$$\frac{h\nu}{kT} = \frac{6.626 \times 10^{-34} \times 4.74 \times 10^{14}}{1.38 \times 10^{-23} \times 370} \approx 61.6$$

The ratio of spontaneous to stimulated emission under thermal equilibrium is:

$$R = e^{h\nu/(kT)} - 1 \approx e^{61.6} \approx 5 \times 10^{26}$$

→ Spontaneous emission is **enormously larger** than stimulated emission in thermal conditions.

Hence, **thermal light cannot cause stimulated emission** — population inversion must be externally created.

Q/ A ruby laser operates at a wavelength of $\lambda = 694.3$ nm and temperature $T = 300$ K. The laser transition occurs between two energy levels with equal degeneracy.

1. Calculate the frequency of the laser transition.
2. Determine the ratio $\frac{h\nu}{kT}$ for this ruby laser.
3. Calculate the ratio R of spontaneous to stimulated emission rates under thermal equilibrium conditions.
4. Calculate the ratio of populations between the upper and lower laser levels $(\frac{N_2}{N_1})$ assuming equal degeneracies.

<u>Solution:</u>

Email: (huda.wasfi@uomus.edu.iq)