Mathematical background; Laplace transform, complex variable, matrices.

Partial Fractions

There are different types of proper fractions, depending on the specific characteristics of the numerator and denominator as follows:

1. Non–repeated Linear Factor

$$\frac{N(s)}{(as+b)(cs+d)} = \frac{A}{(as+b)} + \frac{B}{(cs+d)}$$

Ex.1: If $Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$, find *y(t)*

Solution. We may write $Y(s)$ in terms of its partial-fraction expansion:

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+3}.$$

Using the cover-up method, we get

$$C_1 = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = \frac{8}{3}.$$

In a similar fashion,

$$C_2 = \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -\frac{3}{2}$$

and

$$C_3 = \left.\frac{(s+2)(s+4)}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}.$$

$$y(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

2. Repeated Linear Factor

$$\frac{N(s)}{(as+b)^2} = \frac{A}{(as+b)} + \frac{B}{(as+b)^2}$$

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Ex. 2: If $Y(s) = \frac{(s+3)}{(s+1)(s+2)^2}$, find *y(t)*

Solution. We write the partial fraction as

$$F(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{(s+2)^2}.$$

Then

$$C_1 = (s+1)F(s)|_{s=-1} = \left.\frac{s+3}{(s+2)^2}\right|_{s=-1} = 2,$$

$$C_2 = \frac{d}{ds}(s+2)^2F(s)|_{s=-2} = -2,$$

$$C_3 = (s+2)^2F(s)|_{s=-2} = \left.\frac{s+3}{s+1}\right|_{s=-2} = -1.$$

$$y(t) = 2e^{-t} - 2e^{-2t} - te^{-2t}$$

Ex.3: If $Y(s) = \frac{(s^2+3)}{s(s+2)^2}$, find *y(t)*

Sol. $\frac{(s^2+3)}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{A(s+2)^2+Bs(s+2)+Cs}{s(s+2)^2}$

$(s^2+3) = A(s+2)^2 + Bs(s+2) + Cs$

A=3/4 , B=1/4 , C=-7/2

Then, $\frac{(s^2+3)}{s(s+2)^2} = \frac{3/4}{s} + \frac{1/4}{s+2} - \frac{7/2}{(s+2)^2}$

$$y(t) = \frac{3}{4} + \frac{1}{4}e^{-2t} - \frac{7}{2}te^{-2t}$$

3. Quadratic Factor

$$\frac{(3s^2-4)}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1)+(Bs+C)s}{s(s^2+1)}$$

$$3s^2 - 4 = A(s^2+1) + (Bs+C)s$$

A=4 , B=-1 , C=0

Then, $\frac{(3s^2-4)}{s(s^2+1)} = \frac{4}{s} - \frac{s}{s^2+1}$

$y(t) = 4 - cost$

Compute the inverse Laplace transform of

$$Y(s) = \frac{2s+5}{s^2+8s+12} = \frac{2s+5}{(s+2)(s+6)}$$

Clearly, the two poles ($s = -2, s = -6$) are distinct. Therefore, the partial-fraction expansion is

$$Y(s) = \frac{2s+5}{(s+2)(s+6)} = \frac{a_1}{s+2} + \frac{a_2}{s+6}$$

Using Eq. (8.16), the first residue is

$$a_1 = (s+2)Y(s)|_{s=-2} = \left.\frac{2s+5}{s+6}\right|_{s=-2} = \frac{1}{4} = 0.25$$

$$a_2 = (s+6)Y(s)|_{s=-6} = \left.\frac{2s+5}{s+2}\right|_{s=-6} = \frac{-7}{-4} = 1.75$$

Using the residues, the partial-fraction expansion is

$$Y(s) = \frac{0.25}{s+2} + \frac{1.75}{s+6}$$

$y(t) = 0.25e^{-2t} + 1.75e^{-6t}$

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The Solution of Differential Equations Using Laplace Transforms

The linear differential equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t),$$

$$y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1},$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ and $y_0, y_1, \ldots, y_{n-1}$ are constants, can be solved by Laplace transform techniques. The procedure for solving this IVP is summarized in the following four steps:

1. Take the Laplace Transform of the differential equation.
2. Put initial conditions into the resulting equation.
3. Rearrange your equation to isolate $Y(s)$ equated to something.
4. Calculate the inverse Laplace transform, which will be your final solution to the original differential equation.

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EXAMPLE 1 Solving a First–Order IVP

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t, \qquad y(0) = 6.$$

SOLUTION We first take the transform of the differential equation:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\,\mathscr{L}\{y\} = 13\,\mathscr{L}\{\sin 2t\}$$

$$\mathscr{L}\{dy/dt\} = sY(s) - y(0) = sY(s) - 6,$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4} \qquad \text{or} \qquad (s+3)Y(s) = 6 + \frac{26}{s^2+4}$$

Solving the last equation for $Y(s)$, we get

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}.$$

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}.$$

$$6s^2 + 50 = A(s^2+4) + (Bs+C)(s+3)$$

Setting $s = -3$ then yields $A = 8$

we equate the coefficients of s^2 and s: $\quad 6 = A + B$ and $0 = 3B + C$

$B = -2$, and $\qquad C = 6$

$$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}.$$

$$y(t) = 8\,\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\,\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}.$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

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EXAMPLE 2 Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$

SOLUTION

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

$$y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

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