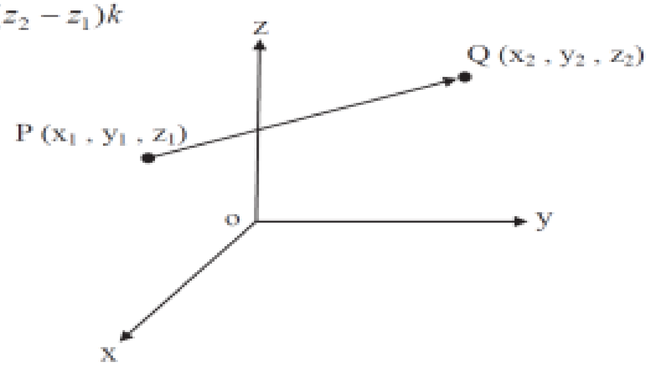
Vector in space

The vector between two points **P** and **Q**

$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$



Algebra of vectors:

Let $V_1 = a_1 i + b_1 j + c_1 k$

$V_2 = a_2 i + b_2 j + c_2 k$

Addition:

$$V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j + (c_1 + c_2)k$$

Subtraction:

$$V_1 - V_2 = (a_1 - a_2)i + (b_1 - b_2)j + (c_1 - c_2)k$$

The magnitude or length of the vector:

$V = ai + bj + ck$

$$|V| = \sqrt{a^2 + b^2 + c^2}$$

Direction of vectors V: $\quad \mathrm{V} = \dfrac{\mathrm{V}}{|\mathrm{V}|}$

Example: find the length of vector $A = i - 2j + 3k$

Solution:

$$|A| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$
$$= \sqrt{1+4+9} = \sqrt{14}$$

Example: find a unit vector u in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution:

$$\overrightarrow{P_1P_2} = (3-1)i + (2-0)j + (0-1)k$$
$$= 2i + 2j - k$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2}$$
$$= \sqrt{4+4+1} = \sqrt{9} = 3$$

$$u = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2i+2j-k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

Example: A force of 6N is applied in the direction of the vector $V = 2i + 2j - k$, express the force as a product of its magnitude and direction

Solution:

The force vector has magnitude 6 and direction $\frac{V}{|V|}$, so:

$$F = 6\frac{V}{|V|} = 6\frac{2i+2j-k}{\sqrt{(2)^2+(2)^2+(-1)^2}} = 6\frac{2i+2j-k}{\sqrt{9}}$$

$$= 6\frac{2i+2j-k}{3}$$

H.W:

1. Let the vector $u = -i + 3j + k$, find $\left|\frac{1}{2}u\right|$

2. find the length and direction of the vector $v = 4i + 3j + 2k$

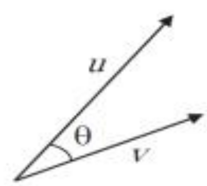
3. find the length of the vector with initial point P(-3,4,1) and terminal point Q(-5,2,2)

The dot product:

Dot product also called scalar products because the resulting products are numbers and not vectors. To calculate ***u.v*** from the component of ***u*** and ***v*** we let:

$$u = a_1 i + b_1 j + c_1 k$$

$$v = a_2 i + b_2 j + c_2 k$$



$$u.v = |u||v|\cos\theta$$

Where θ is the angle between ***u*** and ***v***

$$\cos\theta = \frac{u.v}{|u||v|}$$

$$u.v = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Properties of dot product:

if ***u*** , ***v***, and ***w*** are any vectors and ***c*** is a scalar:

1. $u.v = v.u$

2. $(cu).v = u.(cv) = c(u.v)$

3. $u.(v + w) = u.v + u.w$

4. $u.u = |u|^2$

5. $0.u = 0$

Example: find the angle θ between $A = i - 2j - 2k$ and $B = 6i + 3j + 2k$

Solution:

$$A.B = (1)(6) + (-2)(3) + (-2)(2)$$

$$= 6 - 6 - 4 = -4$$

$$|A| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|B| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{-4}{21} \Longrightarrow \theta = \cos^{-1}\frac{-4}{21} = 100.79^o \approx 101^o$$

Example: find the angle between the vectors $u = 2i + j$, $v = i + 2j - k$

Solution:

$$u.v = (2)(1) + (1)(2) + (0)(-1)$$

$$= 2 + 2 - 0 = 4$$

$$|u| = \sqrt{(2)^2 + (1)^2 + (0)^2} = \sqrt{4+1+0} = \sqrt{5}$$

$$|v| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{u.v}{|u||v|} = \frac{4}{\sqrt{5}\sqrt{6}} = \frac{4}{\sqrt{30}}$$

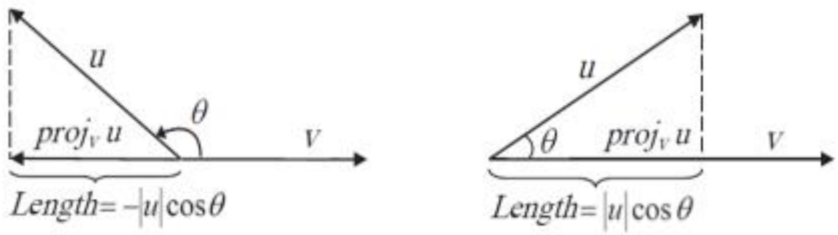
$$\theta = \cos\!-1\frac{4}{\sqrt{30}} = 43.09^o \approx 43^o$$

H.W: find the angle between the vectors $u = 2i - 2j + k$, $v = 3i + 4k$

Vector projections and scalar components:

The vector we get by projecting a vector ***u*** onto the line through a vector ***v*** is called **(the vector projection of u onto v)**, sometimes denoted:

$proj_v u \Longrightarrow$ The vector projection of ***u*** onto ***v***



The vector projection of ***u*** onto ***v*** is the vector:

$$proj_v u = \left(\frac{u.v}{|v|^2}\right) v$$

Where $|v|^2 = v.v$

The scalar component of ***u*** in the direction of ***v*** is the scalar:

$$|u|\cos\theta = \frac{u.v}{|v|} = u.\frac{v}{|v|}$$

Example: find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v

Solution: we find $proj_v u$ from equation:

$$proj_v u = \frac{u.v}{v.v} v = \frac{(6)(1)+(3)(-2)+(2)(-2)}{(1)(1)+(-2)(-2)+(-2)(-2)}(i-2j-2k)$$

$$= \frac{6-6-4}{1+4+4}(i-2j-2k)$$

$$= -\frac{4}{9}(i-2j-2k) = -\frac{4}{9}i + \frac{8}{9}j + \frac{8}{9}k$$

We find the scalar component of ***u*** in the direction of ***v*** from equation

$$|u|\cos\theta = u.\frac{v}{|v|}$$

$$|v| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|u|\cos\theta = (6i+3j+2k).\frac{(i-2j-2k)}{3}$$

$$|u|\cos\theta = (6i+3j+2k).(\frac{1}{3}i - \frac{2}{3}j - \frac{2}{3}k)$$

$$= (6)(\frac{1}{3}) + (3)(-\frac{2}{3}) + (2)(-\frac{2}{3}) = 2 - 2 - \frac{4}{3} = -\frac{4}{3}$$

Example: vectors $u = 5i + 12j$ and $v = \frac{3}{5}i + \frac{4}{5}k$, find:

1. the vector $proj_v u$

2. the scalar component of ***u*** in the direction of ***v***

Solution:

1.

$$proj_v u = \frac{u.v}{v.v} v$$

$$= \frac{(5)(\frac{3}{5}) + (12)(0) + (0)(\frac{4}{5})}{(\frac{3}{5})(\frac{3}{5}) + (0)(0) + (\frac{4}{5})(\frac{4}{5})} (\frac{3}{5}i + \frac{4}{5}k) = \frac{3}{\frac{9}{25} + \frac{16}{25}} (\frac{3}{5}i + \frac{4}{5}k)$$

$$= 3(\frac{3}{5}i + \frac{4}{5}k) = \frac{9}{5}i + \frac{12}{5}k$$

2.

$$|u| \cos\theta = u.\frac{v}{|v|}$$

$$|v| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$

$$|u| \cos\theta = (5i + 12j).(\frac{3}{5}i + \frac{4}{5}k)$$

$$= (5)(\frac{3}{5}) + (12)(0) + (0)(\frac{4}{5})$$

H.W: vectors $u = -2i + 4j - \sqrt{5}k$ and $v = 2i - 4j + \sqrt{5}k$, find:

1. the vector $proj_v u$

2. the scalar component of ***u*** in the direction of ***v***

Cross product:

Two vector ***u*** and ***v*** in space if ***u*** and ***v*** are not parallel, they determine a plane, we select a unit vector ***n*** perpendicular to the plane by the right-hand rule. This means that we choose ***n*** to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from ***u*** to ***v***

Properties of the cross product

if ***u*** , ***v***, and ***w*** are any vectors and ***r*** , ***s*** is are scalars, then:

1. $(ru)\times(sv)=(rs)(u\times v)$

2. $u\times(v+w)=u\times v+u\times w$

3. $v\times u=-(u\times v)$

4. $(v+w)\times u=v\times u+w\times u$

5. $0\times u=0$

6. $u\times(v\times w)=(u.w)v-(u.v)w$

When we apply the definition to calculate the pair wise cross products of ***i*** , ***j***, ***k*** we fined:

$$i\times j=-(j\times i)=k$$
$$j\times k=-(k\times j)=i$$
$$k\times i=-(i\times k)=j$$
$$i\times i=j\times j=k\times k=0$$

Example: find $u\times v$ and $v\times u$ if $u=2i+j+k$ and $v=-4i+3j+k$

Solution:

$$u\times v=\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}i-\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}j+\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}k$$

$$=((1)(1)-(1)(3))i-((2)(1)-(1)(-4))j+((2)(3)-(1)(-4))k$$

$$=(1-3)i-(2+4)j+(6+4)k$$

$$=-2i-6j+10k$$

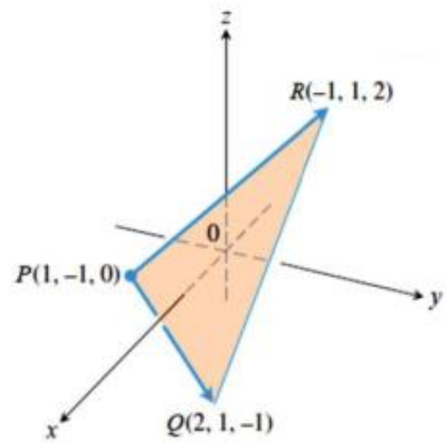
$$v\times u=-(u\times v)=2i+6j-10k$$

Example: find the area of the triangle with vertices p(1,-1,0) , Q(2,1,-1) and R(-1,1,2)

Solution:

The area of the triangle is

$(\frac{1}{2})\ |\overrightarrow{PQ} \times \overrightarrow{PR}|$. in term of components



$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= ((2)(2) - (-1)(2))i - ((1)(2) - (-1)(-2))j + ((1)(2) - (2)(-2))k$$

$$= (4+2)i - (2-2)j + (2+4)k$$

$$= 6i + 6k$$

Hence, the triangle's area is:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6i + 6k| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$$

The triangle's area is half of this $= 3\sqrt{2}$