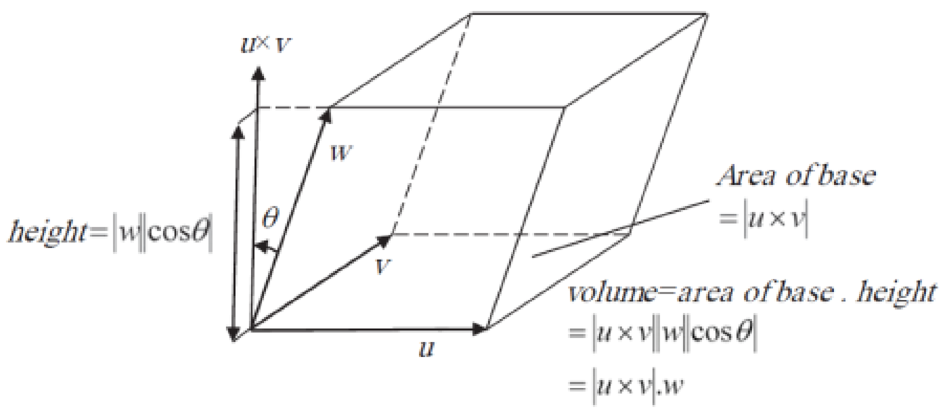
Triple scalar or box product:

The product $(u \times v).w$ is called **the triple scalar product** of ***u***, ***v*** and ***w***.

As you can see from the formula:

$$|(u \times v).w| = |u \times v||w||\cos\theta|$$

The absolute value of this product is the volume of the parallelepiped (parallelogram – side box) determined by ***u***, ***v*** and ***w***. the number $|u \times v|$ is the area of the base parallelogram. The number $|w||\cos\theta|$ is the parallelepiped's height. Because this geometry, $(u \times v).w$ is called **the box product** of ***u***,***v*** and ***w***.



To calculate the triple scalar product:

$$\begin{aligned} \text{Let} \quad u &= a_1 i + b_1 j + c_1 k \\ v &= a_2 i + b_2 j + c_2 k \\ w &= a_3 i + b_3 j + c_3 k \end{aligned}$$

Then

$$(u \times v).w = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example: Find volume of box (parallelepiped) determined by

$$u = i + 2j - k \ , \ v = -2i + 3k \text{ and } w = 7j - 4k$$

Solution:

$$(u \times v).w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix}(1) - \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix}(2) + \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}(-1)$$

$$= [(0)(-4) - (3)(7)](1) - [(-2)(-4) - (3)(0)](2) + [(-2)(7) - (0)(0)](-1)$$

$$= (0 - 21)(1) - (8 - 0)(2) + (-14 - 0)(-1)$$

$$= (-21)(1) - (8)(2) + (-14)(-1)$$

$$= (-21)(1) - (8)(2) + (-14)(-1)$$

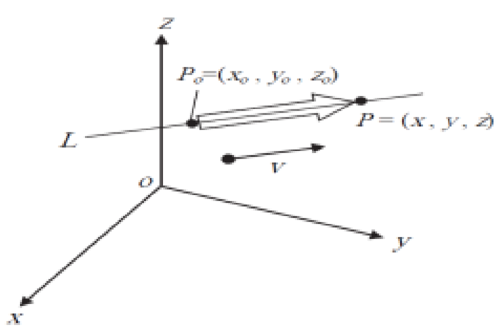
$$= -21 - 16 + 14 = \boxed{-23}$$

The volume is $|(u \times v).w| = 23$ units cubed

H.W: Find volume of box (parallelepiped) determined by $u = i + j - 2k$, $v = -i - k$ and $w = 2i + 4j - 2k$

Equation lines in space:

Suppose that L is a line in space through a point $P_o(x_o , y_o , z_o)$ Parallel to a vector $v = ai + bj + ck$,then L is the set of all points $p\ (x , y , z)$ for which $\overline{P_oP}$ is parallel to v. thus $\overline{P_oP} = tv$ for some scalar parameter t. the value of t depend on the location of the point P along the line. the expanded from of the equation:



$\overrightarrow{P_oP} = tv$

$(x - x_o)i + (y - y_o)j + (z - z_o)k = t(ai + bj + ck)$

$x - x_o = ta$

$y - y_o = tb$

$z - z_o = tc$

From equation above:

The parametric equation for the line through $P_o(x_o , y_o , z_o)$ parallel to $v = ai + bj + ck$ **:**

$$x = x_o + ta$$
$$y = y_o + tb$$
$$z = z_o + tc$$

Example: Find parametric equations for the line through the point $(-2,0,4)$ parallel to the vector $v = 2i + 4j - 2k$

Solution:

With $P_o(x_o , y_o , z_o) = (-2,0,4)$

$x_o = -2 \quad , y_o = 0 \quad , z_o = 4$

and $v = ai + bj + ck = 2i + 4j - 2k$

$a = 2 \quad , b = 4 \quad , c = -2$

$\therefore$

$x = x_o + at \Longrightarrow x = -2 + 2t$

$y = y_o + bt \Longrightarrow y = 4t$

$z = z_o + ct \Longrightarrow z = 4 - 2t$

***Example*:** find parametric equation for the line through the points $P(-3,2,-3)$ and $Q(1,-1,4)$

***Solution*:** the vector

$$\overrightarrow{PQ} = (1-(-3))i + (-1-2)j + (4-(-3))k$$

$$\overrightarrow{PQ} = 4i - 3j + 7k$$

$$\therefore \quad a = 4 \quad , \quad b = -3 \quad , c = 7$$

The vector is parallel to the line with chose ***P*** as the (**base point**):

$$P_o(x_o\ , y_o\ , z_o) \Longrightarrow P(-3,2,-3)$$

$$\therefore \quad x_o = -3 \quad , \quad y_o = 2 \quad , z_o = -3$$

$$\therefore \quad x = x_o + at \Longrightarrow x = -3 + 4t$$

$$y = y_o + bt \Longrightarrow y = 2 - 3t$$

$$z = z_o + ct \Longrightarrow z = -3 + 7t$$

***H.W*:**

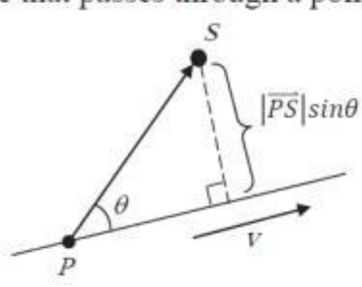
1. Find parametric equation for the line through the point $P(3,-4,-1)$ parallel to the vector $v = i + j + k$

2. Find parametric equation for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$

The distance from a point to a line in space:

To find the distance from a point ***S*** to a line that passes through a point ***P*** parallel to a vector ***v*** :

$$d = \frac{|\overrightarrow{PS} \times v|}{|v|}$$



Example: Find the distance from the point $S(1,1,5)$ to the line

$$L:\ x=1+t \quad ,\ y=3-t \quad ,\ z=2t$$

Solution: from the equations for L

$$L:\ x=1+t \quad ,\ y=3-t \quad ,\ z=2t$$

$$x=x_o+at \ ,\ y_o+bt \ ,\ z=z_o+ct$$

$$\therefore\ x_o=1 \ ,\ y_o=3 \ ,z_o=0 \Longrightarrow \boxed{P(1,3,0)}$$

$$a=1 \ ,\ b=-1 \ ,\ c=2$$

$$and\ \ v=ai+bj+ck \Longrightarrow \boxed{v=i-j+2k}$$

$$\overrightarrow{PS}=(1-1)i+(1-3)j+(5-0)k$$
$$\overrightarrow{PS}=-2j+5k$$

$$and\quad \overrightarrow{PS}\times v=\begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}=\begin{vmatrix} -2 & 5 \\ -1 & 2 \end{vmatrix}i-\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix}j+\begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix}k$$

$$=[(-2)(2)-(5)(-1)]i-[(0)(2)-(5)(1)]j+[(0)(-1)-(-2)(1)]k$$
$$=(-4+5)i-(0-5)j+(0+2)k$$
$$=i+5j+2k$$

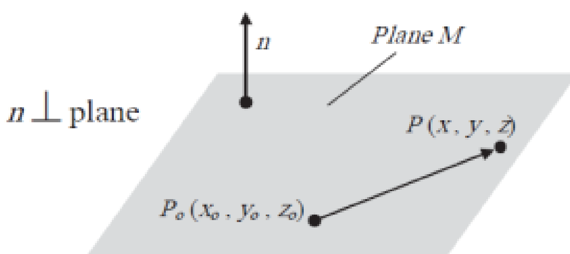
$$d=\frac{|\overrightarrow{PS}\times v|}{|v|}=\frac{\sqrt{(1)^2+(5)^2+(2)^2}}{\sqrt{(1)^2+(-1)^2+(2)^2}}=\frac{\sqrt{1+25+4}}{\sqrt{1+1+4}}=\frac{\sqrt{30}}{\sqrt{6}}=\sqrt{5}$$

H.W: Find the distance from the point $S(0,0,12)$ to the line

$$L:\ x=4t \quad ,\ y=-2t \quad ,\ z=2t$$

Equation for plane in space:

Suppose that plane ***M*** passes through a point $P_o(x_o , y_o , z_o)$ and is normal to the nonzero vector $n = Ai + Bj + Ck$, then ***M*** is the set of all points $P(x,y,z)$ for which $\overrightarrow{P_oP}$ is orthogonal to ***n***. Thus the dot product



$n.\overrightarrow{P_oP} = 0$

This equation is equivalent to:

$(Ai + Bj + Ck).[(x - x_o)i + (y - y_o)j + (z - z_o)k] = 0$

$\boxed{A(x - x_o) + B(y - y_o) + C(z - z_o) = 0}$ ⟹ *Component equation*

This becomes:

$Ax + By + Cz = Ax_o + By_o + Cz_o$

When rearranged or more simply:

$\boxed{Ax + By + Cz = D}$ ⟹ *Component equation simplified*

Where $D = Ax_o + By_o + Cz_o$

Example: Find an equation for the plane through $P_o(-3,0,7)$ perpendicular to $n = 5i + 2j - k$

Solution:

$A = 5 \quad , \quad B = 2 \quad , C = -1$

$x_o = -3 \quad , y_o = 0 \quad , z_o = 7$

$Ax + By + Cz = D$

Where $D = Ax_o + By_o + Cz_o$

$= (5)(-3) + (2)(0) + (-1)(7) = -15 + 0 - 7 = -22$

$\therefore \quad 5x + 2y - z = -22$

Example: Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$

Solution: we find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane:

$$\overrightarrow{AB} = 2i - k$$

$$\overrightarrow{AC} = 3j - k$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} k$$

$$= [(0)(-1) - (-1)(3)]i - [(2)(-1) - (-1)(0)]j + [(2)(3) - (0)(0)]k$$

$$= (0+3)i - (-2-0)j + (6-0)k$$

$$= 3i + 2j + 6k$$

$$\therefore \quad A = 3 \quad , \quad B = 2 \quad , C = 6$$

$$P_o = A(0,0,1) \Longrightarrow x_o = 0 \quad , y_o = 0 \quad , \quad z_o = 1$$

$$\boxed{Ax + By + Cz = D}$$

$$\text{Where} \quad D = Ax_o + By_o + Cz_o$$

$$= (3)(0) + (2)(0) + (6)(1)$$

$$= 0 + 0 + 6 = 6$$

$$\therefore \quad 3x + 2y + 6z = 6$$

As an equation for the plane

Example: Find the point where the line

$x = \frac{8}{3} + 2t \quad , y = -2t \quad , z = 1 + t$ intersects the plane

$3x + 2y + 6z = 6$

Solution:

The point $(\frac{8}{3} + 2t\, , -2t\, , 1 + t)$ lies in the plane if its coordinates satisfy the equation of the plane, that is if:

$$3(\frac{8}{3} + 2t) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$
$$8t + 14 = 6$$
$$8t = 6 - 14$$
$$8t = -8 \quad \Longrightarrow \quad \boxed{t = -1}$$

The point intersection is:

$$(x, y, z)_{t=-1} = (\frac{8}{3} - 2, 2, 1 - 1) = (\frac{2}{3}, 2, 0)$$

H.W:

1. Find an equation for the plane through $P_o(0,2,-1)$ perpendicular to $n = 3i - 2j - k$

2. Find an equation for the plane through $P(1,1,-1)$, $Q(2,0,2)$ and $S(0,-2,1)$

The distance from a point to a plane:

If ***P*** is a point on a plane with normal ***n***, then the distance from any point ***S*** to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto ***n***. that is, the distance from ***S*** to the plane:

$$\boxed{d = \left|\overrightarrow{PS} . \frac{n}{|n|}\right|}$$

Where $n = Ai + Bj + Ck$ is the normal to the plane

Example: Find the distance from $S(1,1,3)$ to the plane $3x+2y+6z=6$

Solution: the distance from ***S*** to the plane is:

$$d = \left|\overrightarrow{PS}.\frac{n}{|n|}\right|$$

From equation of plane $Ax+By+Cz=D$

$$3x+2y+6z=6$$

$$\therefore \quad , \; B=2 \quad , C=6 \; A=3$$

$$n = Ai+Bj+Ck \Longrightarrow \therefore n = 3i+2j+6k$$

The points on the plane easiest to find from the plane's equation are the intercepts. If take P to the y-intercept (0,3,0), then:

$$\overrightarrow{PS} = (1-0)i+(1-3)j+(3-0)k \Longrightarrow \overrightarrow{PS} = i-2j+3k$$

$$|n| = \sqrt{(3)^2+(2)^2+(6)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$$

Note: take P (0, 3, 0)

$$\therefore \; d = \left|(i-2j+3k).\frac{(3i+2j+6k)}{7}\right|$$

$$= \left|(i-2j+3k).(\frac{3}{7}i+\frac{2}{7}j+\frac{6}{7}k)\right|$$

$$= \left|(1)(\frac{3}{7})+(-2)(\frac{2}{7})+(3)(\frac{6}{7})\right|$$

$$= \left|\frac{3}{7}-\frac{4}{7}+\frac{18}{7}\right| = \frac{17}{7}$$

H.W:

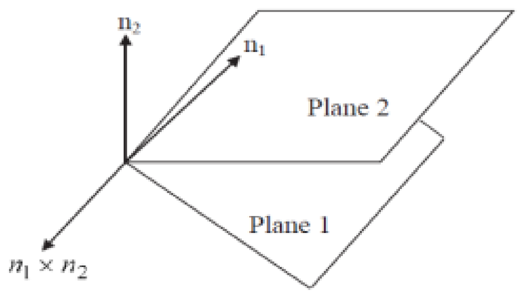
1. Find the distance from the point $S(2,-3,4)$ to the plane $x+2y+2z=13$
Take P (1, 2, 1).

2. Find the distance from the point $S(0,-1,0)$ to the plane $2x+y+2z=13$
Take P (0, -1, 0).

Angle between planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors.

$$\theta = \cos^{-1}\left(\frac{n_1 . n_2}{|n_1||n_2|}\right)$$



Example: Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$

Solution: from equation of plane $Ax + By + Cz = D$

$$n = Ai + Bj + Ck$$

For plane 1: $A = 3 \quad , \quad B = -6 \quad , C = -2$

$$\therefore \boxed{n_1 = 3i - 6j - 2k}$$

For plane 2: $A = 2 \quad , \quad B = 1 \quad , C = -2$

$$\therefore \boxed{n_2 = 2i + j - 2k}$$

are normal to the planes. The angle between them is:

$$\theta = \cos^{-1}\left(\frac{n_1 . n_2}{|n_1||n_2|}\right)$$

$$n_1.n_2 = (3i - 6j - 2k).(2i + j - 2k)$$
$$n_1.n_2 = (3)(2) + (-6)(1) + (-2)(-2) = 6 - 6 + 4 = \boxed{4}$$

$$|n_1| = \sqrt{(3)^2 + (-6)^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = \boxed{7}$$

$$|n_2| = \sqrt{(2)^2 + (1)^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = \boxed{3}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^o$$

H.W: find the angle between the planes $x + y = 1$ and $2x + y - 2z = 2$

Thanks for lessening ..

Any questions?