Bisection Method

The bisection method is an approximation method to find the roots of the given equation by repeatedly dividing the interval. This method will divide the interval until the resulting interval is found, which is extremely small.

Steps for Bisection Method:

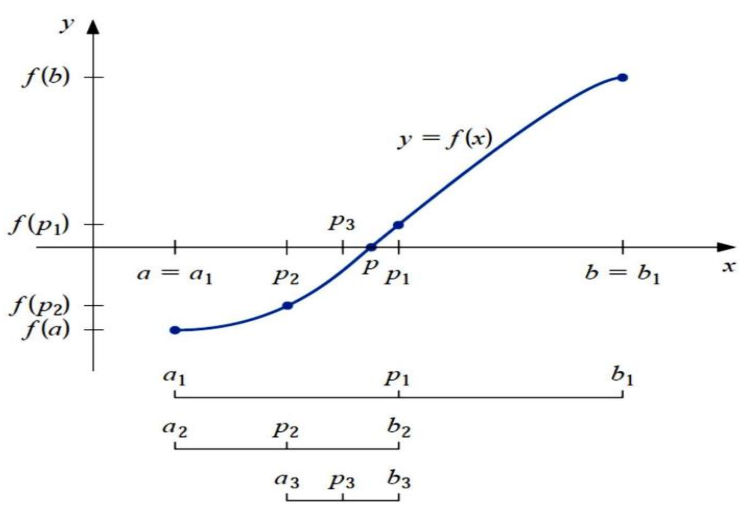
Let f be continuous function on the closed interval $[a, b]$, and assume that the condition $(a)f(b) < 0$ is satisfied, we follow the following steps:

Step 1: We bisect the space as follows: $c = \frac{a+b}{2}$ or $c_n = \frac{a_n+b_n}{2}$

Step 2: If $f(c) = 0$, it follows that c is the exact root, otherwise check the sign of $f(c)$, if $f(c) < 0, f(b) > 0$, then the exact root belong to $[c, b]$, and we set $a = c, b = b$, and then we go to Step 1.

While, if $(c) > 0, f(a) < 0$, then the exact root belong to $[a, c]$, And we set $a = a$, $b = c$, and then we go to Step 1.

Step 3: We continue iteratively, until we get the following condition is satisfied $|(c_n)| < \epsilon, or \quad |c_{n+1} - c_n| < \epsilon, or \ |b_n - a_n| < \epsilon$



Example: consider that, we have the following equation

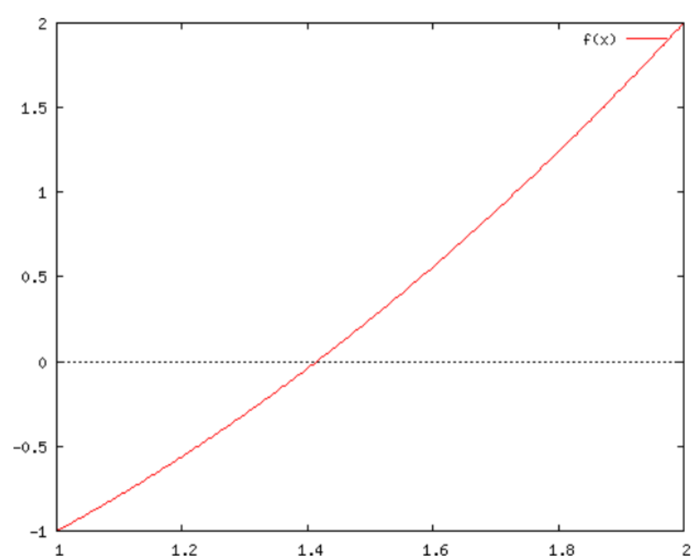
$$(x) = x^2 - 2 = 0, x \in [1, 2],$$

1- Compute the approximate roots of this equation, with using Bisection algorithm, for three iterative steps.
2- Find the iterative errors as each step.
3- Since the exact solution is known for this equation, find also the absolute errors at each step.

Solution:

Since $(1) = -1 < 0$ *and* $f(2) = 2 > 0$, there exists a root in the interval [1, 2]

$x^2 - 2 = 0 \rightarrow \quad x^2 = 2 \rightarrow x = \mp\sqrt{2}$, Clearly, $\alpha = \sqrt{2} \approx 1.414$, is the exact root of f in the interval [1, 2]



When n=1 , the interval is [1, 2]

$a_1 = 1, \quad b_1 = 2, \; c_1 = \frac{a_1+b_1}{2} = 1.5, \quad f(c_1) = f(1.5) = 0.25 > 0$

we had $(1) < 0$ *and* $f(2) > 0$ and we got $f(1.5) > 0$, so we have that:

When n=2, the new interval is [1, 1.5]

$a_2 = 1, \quad {}_2 = 1.5,\ c_2 = \frac{a_2+b_2}{2} = 1.25,\ f(c_2) = f(1.25) = -0.4375 < 0$

we had $(1) < 0$ *and* $f(1.5) > 0$ and we got $f(1.25) < 0$, so we have that:

When n=3, the new interval is [1.25, 1.5]

$a_3 = 1.25, \quad {}_3 = 1.5,\ c_3 = \frac{a_3+b_3}{2} = \frac{1.25+1.5}{2} = 1.375$

The iterative errors can be found as follows: $E_n = |c_{n+1} - c_n|$,

$$E_1 = |c_2 - c_1| = |1.25 - 1.5| = 0.25$$

$$|E_2 = |c_3 - c_2| = |1.375 - 1.25| = 0.125$$

clearly, it is, still too large, so we have to continue in the iterative processes until we get the convergent condition, $|c_{n+1} - c_n| < \epsilon$, is satisfied . or $|b_n - a_n| < \epsilon$

The Absolute errors can be found as follows:

$$\text{Absolute error} = |c_n - \alpha|$$

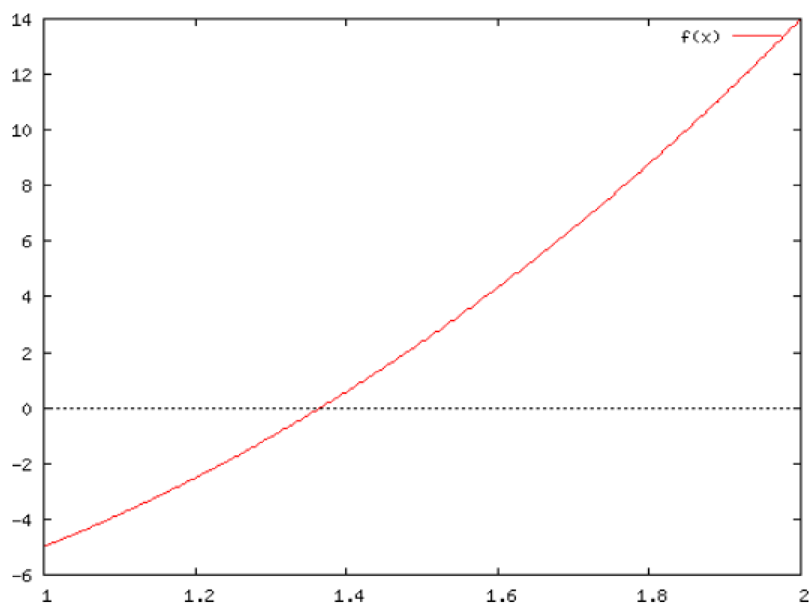
n	a_n	b_n	c_n	$f(c_n)$	E_n $\lvert c_{n+1} - c_n \rvert$	Absolute Errors $\lvert c_n - 1.414 \rvert$
1	1	2	1.5	0.25	0.25	0. 086
2	1	1.5	1.25	-0.4375	0.125	0.164
3	1.25	1.5	1.375	-0.1094		0.039

Example:

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least $\epsilon = 0.00013$.

Solution:

Since $(1) = -5 < 0$ *and* $f(2) = 14 > 0$, there exists a root in the interval [1, 2]



Clearly, $\alpha \approx 1.365230013$, is the approximate root of f in the interval [1, 2]

When n=1 , the interval is [1, 2]

$a_1 = 1, \quad {}_1 = 2, \; c_1 = \frac{a_1+b_1}{2} = 1.5, \quad f(c_1) = f(1.5) = 2.375 > 0$

we had $(1) < 0$ *and* $f(2) > 0$ and we got $f(1.5) > 0$, so we have that:

When n=2, the new interval is [1, 1.5]

$a_2 = 1, \quad {}_2 = 1.5, \; c_2 = \frac{a_2+b_2}{2} = 1.25, \; f(c_2) = f(1.25) = -0.7969 < 0$

we had $(1) < 0$ *and* $f(1.5) > 0$ and we got $f(1.25) < 0$, so we have that:

When n=3, the new interval is [1.25, 1.5]

$a_3 = 1.25, \quad {}_3 = 1.5, \; c_3 = \frac{a3+b3}{2} = \frac{1.25+1.5}{2} = 1.375$

Continuing in this manner gives the values in the table below:

n	a_n	b_n	$c_i p_n$	$f(c_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

we had $(a_{13}) = f(1.364990235) < 0$ *and* $f(b_{13}) = f(1.365234375) > 0$ and we got $f(c_{13}) = f(1.365112305) < 0$, so we have that:

When n=14, the new interval is $[a_{14}, b_{14}]$ =[1.365112305, 1.365234375]

We calculate the error using the following: $|b_n - a_n| < \epsilon$

Hence, : $|b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070 < \epsilon$

Theorem:

Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when} \quad n \ge 1.$$

Example:

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Solution:

$$|p_n - p| \le \frac{b-a}{2^n} < 10^{-3}$$

$$\rightarrow \frac{2-1}{2^n} < 10^{-3} \quad \rightarrow \quad \frac{1}{2^n} < 10^{-3} \quad \rightarrow \quad 2^{-n} < 10^{-3} \rightarrow \log_{10} 2^{-n} < \log_{10} 10^{-3}$$

$$\rightarrow -n \log_{10} 2 < -3 \log_{10} 10 \quad \rightarrow \quad n \log_{10} 2 > 3 \log_{10} 10 \qquad \log_{10} 10 = 1$$

$$\rightarrow (0.301) > 3 \quad \rightarrow n > \frac{3}{0.301} \approx 9.966 \rightarrow n = 10 \qquad \log_{10} 2 \approx 0.301$$

Exercises:

Q1: Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Q2: Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.

a. $[0, 1]$ **b.** $[1, 3.2]$ **c.** $[3.2, 4]$

Q3: Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.

a. $x - 2^{-x} = 0$ for $0 \le x \le 1$

b. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$

c. $2x\cos(2x) - (x+1)^2 = 0$ for $-3 \le x \le -2$ and $-1 \le x \le 0$

d. $x\cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$