(1) Gauss Elimination Method :

To reduce the augmented matrix to row - echelon form you should follow the following steps:

Step 1. Locate the leftmost column that does not consist entirely of zeros.

Step 2. Interchange the top row with another row , if necessary , to bring a nonzero entry to the top of the column found in Step 1.

Step 3. If the entry that is now at the top of the column found in Step 1 is b , multiply the first row by $\frac{1}{b}$ in order to introduce a leading 1.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row - echelon form.

Example 3.2.8: Solve the following system of linear equations by using the Gauss elimination method:

$$5x_1 + 6x_2 = 7$$
$$3x_1 + 4x_2 = 5$$

Solution: The system of linear equations has the following augmented matrix

$$\left(\begin{array}{cc|c} 5 & 6 & 7 \\ 3 & 4 & 5 \end{array}\right) \xrightarrow{\frac{1}{5}R_1 \rightarrow R_1} \left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 3 & 4 & 5 \end{array}\right) \xrightarrow{R_2 - 3R_1 \rightarrow R_2}$$

$$\left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 0 & \frac{2}{5} & \frac{4}{5} \end{array}\right) \xrightarrow{\frac{5}{2}R_2 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 0 & 1 & 2 \end{array}\right)$$

In Row - Echelon Form

The last matrix is in row - echelon form . The corresponding reduced system is:

$$x_1 + \frac{6}{5}x_2 = \frac{7}{5} \quad \dots (1)$$

$$x_2 = 2 \quad \dots (2)$$

Substitute the value of x_2 in equation (1) ,we get

$$x_1 + \frac{12}{5} = \frac{7}{5} \Rightarrow x_1 = \frac{7}{5} - \frac{12}{5} \Rightarrow x_1 = -\frac{5}{5} = -1$$

Therefore the solution of the system is $x_1 = -1$, and $x_2 = 2$.

Example 3.2.9: Solve the following system of linear equations by using the Gauss elimination method :

$$\begin{aligned} 4y + 2z &= 1 \\ 2x + 3y + 5z &= 0 \\ 3x + y + z &= 11 \end{aligned}$$

Solution: The system of linear equations has the following augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 4 & 2 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 1 & 1 & 11 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 3 & 1 & 1 & 11 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 4 & 2 & 1 \\ 3 & 1 & 1 & 11 \end{array}\right) \xrightarrow{R_3 - 3R_1 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 4 & 2 & 1 \\ 0 & -\frac{7}{2} & -\frac{13}{2} & 11 \end{array}\right) \xrightarrow{\frac{1}{4}R_2 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{7}{2} & -\frac{13}{2} & 11 \end{array}\right) \xrightarrow{R_3 + \frac{7}{2}R_2 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & -\frac{19}{4} & \frac{95}{8} \end{array}\right) \xrightarrow{-\frac{4}{19}R_3 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right)$$

In Row - Echelon Form

The last matrix is in row - echelon form . The corresponding reduced system is:

$$x + \frac{3}{2}y + \frac{5}{2}z = 0 \quad \dots (1)$$

$$y + \frac{1}{2}z = \frac{1}{4} \quad \dots (2)$$

$$z = -\frac{5}{2} \quad \dots (3)$$

Substitute the value of z in equation (2) , we get

$$y - \frac{5}{4} = \frac{1}{4} \Rightarrow y = \frac{1}{4} + \frac{5}{4} = \frac{6}{4} = \frac{3}{2}$$

Substitute the values of y and z in equation (1) , we get

$$x + \frac{9}{4} - \frac{25}{4} = 0 \quad \Rightarrow \quad x = -\frac{9}{4} + \frac{25}{4} = \frac{16}{4} = 4$$

Therefore the solution of the system is $x = 4$, $y = \frac{3}{2}$, and $z = -\frac{5}{2}$.

Example 3.2.10: Solve the following system of linear equations by using the Gauss elimination method:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

Solution: The system of linear equations has the following augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ -2 & -3 & 4 & -6 \end{array}\right) \xrightarrow{R_3 + 2R_1 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In Row – Echelon Form

The last matrix is in row-echelon form. The corresponding reduced system is:

$$x_1 + 2x_2 - 3x_3 = 5 \quad \ldots(1)$$
$$x_2 - 2x_3 = 4 \quad \ldots(2)$$

Equation (2) implies that $x_2 = 2x_3 + 4 \quad \ldots(3)$

Substitute the value of x_2 in equation (1), we get

$$x_1 + 4x_3 + 8 - 3x_3 = 5 \quad \Rightarrow \quad x_1 = -x_3 - 3$$

If we let $x_3 = t$, then for any real number t , we have:

$$x_1 = -t - 3$$
$$x_2 = 2t + 4$$
$$x_3 = t$$

Therefore the system has infinite number of solutions.

If t = 0, the solution will be $x_1 = -3$, $x_2 = 4$ and $x_3 = 0$

If t = 1, the solution will be $x_1 = -4$, $x_2 = 6$ and $x_3 = 1$

If t = −2, the solution will be $x_1 = -1$, $x_2 = 0$ and $x_3 = -2$

If t = 3.5, the solution will be $x_1 = -6.5$, $x_2 = 11$ and $x_3 = 3.5$

Exercises 3.2.11:

(1) Solve the following system of linear equations by using the Gauss elimination method:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= 9 \\ x_1 + 3x_2 + 2x_3 &= 5 \end{aligned}$$

(2) Solve the following system of linear equations by using the Gauss elimination method:

$$\begin{aligned} x + y + 2z &= 14 \\ x - 3y + 2z &= 10 \\ 2x - y + 2z &= 15 \end{aligned}$$