FIXED POINT ITERATION

In this section, we consider an important iterative method for the solution of equations of the form

$$g(x) = x. \tag{1}$$

A solution to this equation is called a **fixed point** of g. The fixed point iteration, which is used to determine the fixed point of g, starts from an initial guess p_0 to determine a sequence of values $\{p_n\}$ obtained from the rule

$$p_{n+1} = g(p_n), \qquad n = 0, 1, 2, \ldots. \tag{2}$$

If the process converges to a root α and g is continuous, then

$$\lim_{n\to\infty} p_n = \alpha$$

and a solution to Eqn. (1) is obtained.
Geometrically, a fixed point p of g can be regarded as the point of intersection of the functions $y = x$ and $y = g(x)$. Figures 1 and 2 give graphical interpretations of convergence and divergence iterations, respectively.

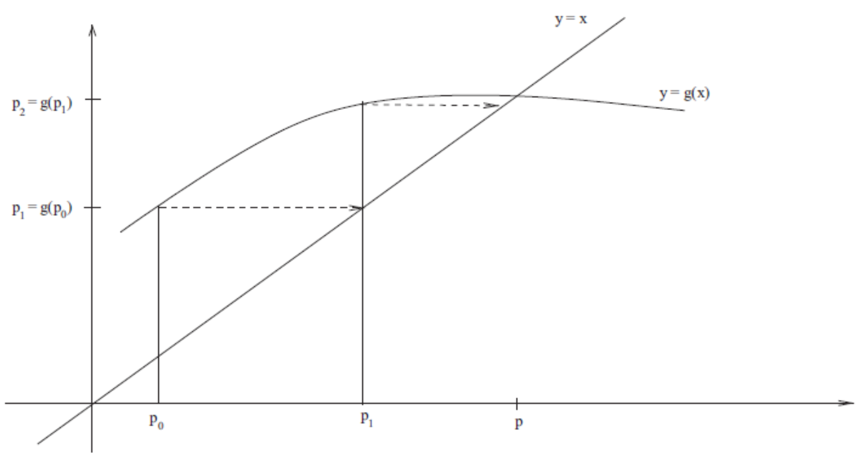

FIGURE 1
The fixed point iteration with convergence iteration to its zero p.

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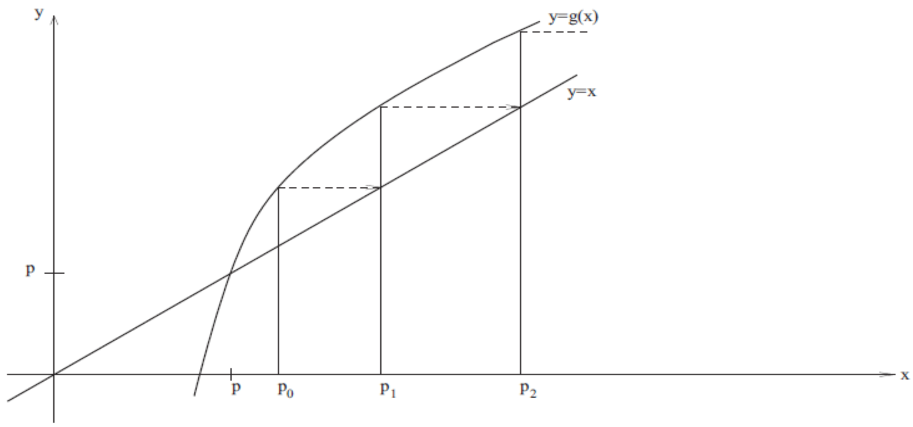

FIGURE 2
The fixed point iteration with divergence iteration from its zero p.

EXAMPLE 1

Solve the equation $0.4 - 0.1x^2 = 0$ with $x_0 = 1$ using the fixed point iteration and choosing

(F1) $\quad g(x) = 0.4 + x - 0.1x^2,$

(F2) $\quad g(x) = \frac{4}{x}.$

Table 1 gives the iterates x_n of the fixed point iteration method for both F1 and F2. The exact solution is 2.0. From the results given in Table 1, it is interesting to note that the choice of $g(x)$ in F1 led to convergence and the one in F2 to divergence.

The question that one can ask is how can we choose $g(x)$ so that convergence will occur? The following theorem gives an answer to that question.

THEOREM Fixed Point Theorem

If g is continuous on $[a, b]$ and $a \le g(x) \le b$ for all x in $[a, b]$, then g has at least a fixed point in $[a, b]$.

Further, suppose $g'(x)$ is continuous on (a, b) and that a positive constant c exists with

$$|g'(x)| \le c < 1, \qquad \text{for all } x \text{ in } (a, b). \tag{3}$$

Then there is a unique fixed point α of g in $[a, b]$. Also, the iterates

$$x_{n+1} = g(x_n) \qquad n \ge 0$$

will converge to α for any choice of x_0 in $[a, b]$.

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n	F1	F2
0	1.00000000	1
1	1.30000000	4
2	1.53100000	1
3	1.69660390	4
4	1.80875742	1
5	1.88159708	4
6	1.92755632	1
7	1.95600899	4
8	1.97341187	1
9	1.98397643	4
10	1.99036018	1
11	1.99420682	4
12	1.99652073	1
13	1.99791123	4
14	1.99874630	1
15	1.99924762	4
16	1.99954852	1
17	1.99972909	4
18	1.99983745	1

Table 1 The fixed point iteration for (F1) and (F2).

EXAMPLE 2

Suppose we want to solve the equation $x = \pi + 0.5\sin x$ using the fixed point iteration. If $g(x) = \pi + 0.5\sin x$, let us find the interval $[a, b]$ satisfying the hypotheses of Theorem

It is easy to see that g is continuous and

$$0 \le g(x) \le 2\pi \qquad \text{for all } x \text{ in } [0, 2\pi].$$

Moreover, g' is continuous and

$$|g'(x)| = |0.5\cos x| < 1 \text{ in } [0, 2\pi];$$

thus, g satisfies the hypotheses of Theorem and has a unique fixed point in the interval $[0, 2\pi]$.

EXAMPLE 3

The equation $x = \frac{5}{x^2} - 2$ has exactly one zero in $[2.5, 3]$. Use the fixed point method to find the root with $g(x) = 5/x^2 + 2$ and $x_0 = 2.5$ to within 10^{-4}.

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Both $g(x)$ and $g'(x) = -10/x^3$ are continuous in $[2.5, 3]$. It is easy to check that $|g'(x)| \leq 1$ and $2.5 \leq g(x) \leq 3$ for all x in $[2.5, 3]$. Thus, g has a unique fixed point in $[2.5, 3]$.

The first three iterations are

$$p_1 = 5/p_0^2 - 2 = 2.8$$
$$p_2 = 5/p_1^2 - 2 = 2.637755$$

$p_3 = 5/P_2^2 - 2 = 2.718623.$

Continuing in this manner leads to the values in Table 2 , which converge to $r = 2.69062937$.

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» fixed('g1',2.5,10^(-4),100)
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iter	xn	g(xn)	| xn+1-xn |
0	2.500000	2.800000	
1	2.800000	2.637755	0.30000000
2	2.637755	2.718623	0.16224490
3	2.718623	2.676507	0.08086781
4	2.676507	2.697965	0.04211628
5	2.697965	2.686906	0.02145790
6	2.686906	2.692572	0.01105819
7	2.692572	2.689660	0.00566568
8	2.689660	2.691154	0.00291154
9	2.691154	2.690387	0.00149391
10	2.690387	2.690781	0.00076713
11	2.690781	2.690579	0.00039377
12	2.690579	2.690683	0.00020216
13	2.690683	2.690629	0.00010378
14	2.690629	2.690657	0.00005328

Table 3.3 Solution of $x = \frac{5}{x^2} + 2$ using the fixed point iteration method with $x_0 = 2.5$.

EXERCISE

1. Use the fixed point iteration to find an approximation to $\sqrt{2}$ with 10^{-3} accuracy.

2. The quadratic equation $x^2 - 2x - 3 = 0$ has two roots. Consider the following rearrangements to approximate the roots using the fixed point iteration:

   (a) $x = \sqrt{2x + 3}$,
   (b) $x = 3/(x - 2)$,
   (c) $x = (x^2 - 3)/2$,

   starting from $x_0 = 4$.

3. Solve the following equations using the fixed point iteration method:

   (a) $x = \sin x + x + 1$, in $[3.5, 5]$,
   (b) $x = \sqrt{x^2 + 1} - x + 1$, in $[0, 3]$,
   (c) $x = \ln x^2 + x - 2$, in $[-4, -2]$.

4. For each of the following functions, locate an interval on which fixed point iteration will converge.

   (a) $x = 0.2 \sin x + 1$,
   (b) $x = 1 - x^2/4$.

5. Find the solution of the following equations using the fixed point iteration:

   (a) $x = x^5 - 0.25$ starting from $x_0 = 0$,
   (b) $x = 2 \sin x$ starting from $x_0 = 2$,
   (c) $x = \sqrt{3x + 1}$ starting from $x_0 = 2$,
   (d) $x = \frac{2 - e^x + x^2}{3}$ starting from $x_0 = 1$.

6. Let $g(x) = \frac{x+4}{2}$.

   (a) Show that $\alpha = 4$ is a fixed point of $g(x)$,
   (b) Let $x_0 = 5$, show that $|\alpha - x_n| = |\alpha - x_0|/2^n$ for $n = 1, 2 \ldots.$

THE SECANT METHOD

Because the bisection and the false position methods converge at a very slow speed, our next approach is an attempt to produce a method that is faster. One such method is the **secant method**. Similar to the false position method, it is based on approximating the function by a straight line connecting two points on the graph of the function f, but we do not require f to have opposite signs at the initial points. Figure 3.7 illustrates the method.

In this method, the first point, x_2, of the iteration is taken to be the point of intersection of the x-axis and the secant line connecting two starting points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. The next point, x_3, is generated by the intersection of the new secant line, joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ with the x-axis. The new point, x_3, together with x_2, is used to generate the next point, x_4, and so on.

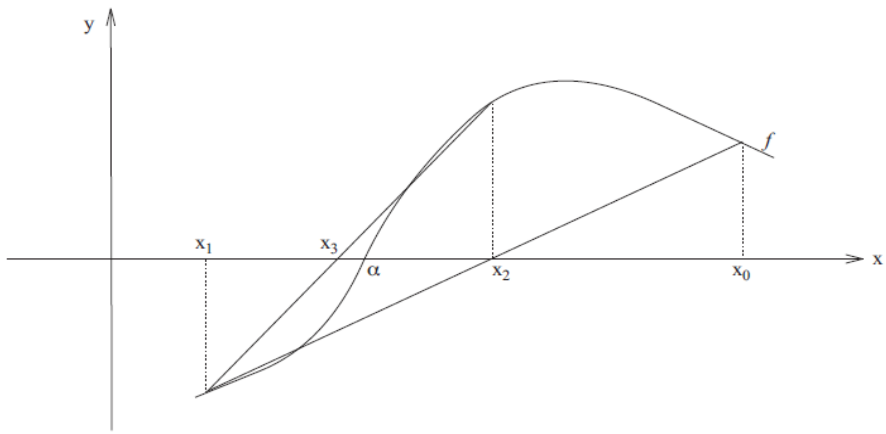

FIGURE 3
The secant method and the first two approximations to its zero α.

A formula for x_{n+1} is obtained by setting $x = x_{n+1}$ and $y = 0$ in the equation of the secant line from $(x_{n-1}, f(x_{n-1}))$ to $(x_n, f(x_n))$

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]. \tag{4}$$

Note that x_{n+1} depends on the two previous elements of the sequence and therefore two initial guesses, x_0 and x_1, must be provided to generate $x_2, x_3, \ldots$.

An algorithmic statement of this method is shown below.

Let x_0 and x_1 be two initial approximations,

for $n = 1, 2, \ldots, \text{ITMAX}$

$\lfloor \quad x_{n+1} \leftarrow x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right].$

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A suitable stopping criterion is

$$|f(x_n)| \leq \epsilon, \quad |x_{n+1} - x_n| \leq \epsilon \quad \text{or} \quad \frac{|x_{n+1} - x_n|}{|x_{n+1}|} \leq \epsilon$$

where ϵ is a specified tolerance value.

EXAMPLE 4

Use the secant method with $x_0 = 1$ and $x_1 = 2$ to solve $x^3 - x^2 - 1 = 0$ to within 10^{-4}.

With $x_0 = 1$, $f(x_0) = -1$ and $x_1 = 2$, $f(x_1) = 3$, we have

$$x_2 = 2 - \frac{(2-1)(3)}{3-(-1)} = 1.25$$

from which $f(x_2) = f(1.25) = -0.609375$. The next iterate is

$$x_3 = 1.25 - \frac{(1.25-2)(-0.609375)}{-0.609375 - 3} = 1.3766234.$$

Continuing in this manner leads to the values in Table 3 which converge to $r = 1.4655713$.

EXERCISE

1. Approximate to within 10^{-6} the root of the equation $e^{-2x} - 7x = 0$ in $[1/9, 2/3]$ by the secant method.

» secant('f1',1,2,10^(-4),40)

n	xn	f(xn)	f(xn+1)-f(xn)	| xn+1-xn |
0	1.000000	-1.000000		
1	2.000000	3.000000	4.000000	1.000000
2	1.250000	-0.609375	-3.609375	0.750000
3	1.376623	-0.286264	0.323111	0.126623
4	1.488807	0.083463	0.369727	0.112184
5	1.463482	-0.007322	-0.090786	0.025325
6	1.465525	-0.000163	0.007160	0.002043
7	1.465571	3.20E-07	0.000163	0.000046

Table 3 Solution of $x^3 - x^2 - 1 = 0$ using the secant method with $x_0 = 1$, $x_1 = 2$.

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2. The function $f(x) = x^5 - 3x^3 + 2.5x - 0.6$ has two zeros in the interval $[0, 1]$. Approximate these zeros by using the secant method with:

   (a) $x_0 = 0, x_1 = 1/2$,

   (b) $x_0 = 3/4, x_1 = 1$.

3. Solve the equation $x^3 - 4x^2 + 2x - 8 = 0$ with an accuracy of 10^{-4} by using the secant method with $x_0 = 3, x_1 = 1$.

4. Use the secant method to approximate the solution of the equation $x = x^2 - e^{-x}$ to within 10^{-5} with $x_0 = -1$ and $x_1 = 1$.

5. If the secant method is used to find the zeros of $f(x) = x^3 - 3x^2 + 2x - 6$ with $x_0 = 1$ and $x_1 = 2$, what is x_2?

6. Use the secant method to approximate the solution of the equation $x = -e^x \sin x - 5$ to within 10^{-10} in the interval $[3, 3.5]$.

7. Given the following equations:

   (a) $x^4 - x - 10 = 0$,

   (b) $x - e^{-x} = 0$.

   Determine the initial approximations. Use these to find the roots correct to four decimal places using the secant method.

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