



Indeterminate Forms and L'Hôpital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or ∞ . The rule is known today as **L'Hôpital's Rule**

THEOREM L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

EXAMPLE 1 Using L'Hôpital's Rule

$$(a) \lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$$

$$(b) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \frac{\frac{1}{2\sqrt{1+x}}}{1} \Big|_{x=0} = \frac{1}{2}$$

EX-1 – Evaluate the following limits :

$$1) \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$2) \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4}$$

$$3) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

$$4) \lim_{x \rightarrow \frac{\pi}{2}} - \left(x - \frac{\pi}{2}\right) \cdot \tan x$$



Sol. –

$$1) \lim_{x \rightarrow 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \text{ u sin g L' Hoptal' s rule} \Rightarrow \\ = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2) \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4} \Rightarrow \frac{0}{0} \text{ u sin g L' Hoptal' s rule} \Rightarrow \\ = \lim_{x \rightarrow 2} \frac{x}{\sqrt{x^2 + 5}} = \lim_{x \rightarrow 2} \frac{1}{2\sqrt{x^2 + 5}} = \frac{1}{2\sqrt{4 + 5}} = \frac{1}{6}$$

$$3) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \Rightarrow \frac{0}{0} \text{ u sin g L' Hoptal' s rule} \Rightarrow \\ = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \Rightarrow \frac{0}{0} \text{ u sin g L' Hopital' s rule} \Rightarrow \\ = \frac{1}{6} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{6}$$

$$4) \lim_{x \rightarrow \frac{\pi}{2}} - \left(x - \frac{\pi}{2}\right) \tan x \Rightarrow 0 \cdot \infty \text{ we can't u sin g L' Hoptal' s rule} \Rightarrow \\ = \lim_{x \rightarrow \frac{\pi}{2}} - \frac{x - \frac{\pi}{2}}{\cos x} \cdot \lim_{x \rightarrow \frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \text{ u sin g L' Hopital' s rule} \Rightarrow \\ = \lim_{x \rightarrow \frac{\pi}{2}} - \frac{1}{-\sin x} \cdot \lim_{x \rightarrow \frac{\pi}{2}} \sin x = \frac{1}{\sin \frac{\pi}{2}} \cdot \sin \frac{\pi}{2} = 1$$



The slope of the curve

Secant to the curve is a line through two points on a curve. Slopes and tangent lines:

1. we start with what we can calculate , namely the slope of secant through P and a point Q nearby on the curve .
2. we find the limiting value of the secant slope (if it exists) as Q approaches p along the curve .
3. we take this number to be the slope of the curve at P and define the tangent to the curve at P to be the line through p with this slope .

The derivative of the function f is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

EX-2- Write an equation for the tangent line at $x = 3$ of the curve :

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol.-

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2 * 3 + 3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x - 3) \Rightarrow 27y + x = 12$$



Velocity and acceleration and other rates of changes

The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t .

i.e

$$v = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a , then its velocity after time t is $v + at$.

$$\text{average acceleration} = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e.

$$a = \frac{dv}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$$



The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$

$$\text{average rate of change} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f' = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

EX-1- The position s (in meters) of a moving body as a function of time t (seconds) is : $s = 2t^2 + 5t - 3$; find :

a) The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .

b) The body's velocity at $t = 2$ seconds .

Sol.-

$$\begin{aligned} \text{a) 1) } \Delta s &= s(t + \Delta t) - s(t) = 2(t + \Delta t)^2 + 5(t + \Delta t) - 3 - [2t^2 + 5t - 3] \\ &= (4t + 5)\Delta t + 2(\Delta t)^2 \end{aligned}$$

$$\text{at } t = 0 \text{ and } \Delta t = 2 \Rightarrow \Delta s = (4 * 0 + 5) * 2 + 2 * 2^2 = 18$$

$$\text{2) } v_{av} = \frac{\Delta s}{\Delta t} = \frac{(4t + 5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2\Delta t$$

$$\text{at } t = 0 \text{ and } \Delta t = 2 \Rightarrow v_{av} = 4 * 0 + 5 + 2 * 2 = 9$$

$$\text{b) } v(t) = \frac{d}{dt} f(t) = 4t + 5$$

$$v(2) = 4 * 2 + 5 = 13$$



EX-4- A particle moves along a straight line so that after t (seconds) , its distance from O a fixed point on the line is s (meters) , where: $s = t^3 - 3t^2 + 2t$

- i) when is the particle at O ?
- ii) what is its velocity and acceleration at these times ?
- iii) what is its average velocity during the first second ?
- iv) what is its average acceleration between $t = 0$ and $t = 2$?

Sol. –

i) at $s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$
either $t = 0$ or $t = 1$ or $t = 2$ sec.

ii) velocity = $v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m / s$
 $\Rightarrow v(1) = -1m / s$
 $\Rightarrow v(2) = 2m / s$

acceleration = $a(t) = 6t - 6 \Rightarrow a(0) = -6m / s^2$
 $\Rightarrow a(1) = 0m / s^2$
 $\Rightarrow a(2) = 6m / s^2$

iii) $v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(1) - s(0)}{1 - 0} = \frac{1 - 3 + 2 - 0}{1} = 0m / s$

iv) $a_{av} = \frac{\Delta v}{\Delta t} = \frac{v(2) - v(0)}{2 - 0} = \frac{2 - 2}{2} = 0m / s^2$



Exercises

1-Find the limits for the following functions by using L'Hopital's rule :

$$1) \lim_{x \rightarrow \infty} \frac{5x^2 - 3x}{7x^2 + 1}$$

$$2) \lim_{t \rightarrow 0} \frac{\sin t^2}{t}$$

$$3) \lim_{x \rightarrow \frac{\pi}{2}} \frac{2x - \pi}{\cos x}$$

$$4) \lim_{t \rightarrow 0} \frac{\cos t - 1}{t^2}$$

$$5) \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x}$$

$$6) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$7) \lim_{x \rightarrow 1} \frac{2x^2 - (3x + 1)\sqrt{x + 2}}{x - 1}$$

$$8) \lim_{x \rightarrow 0} \frac{x(\cos x - 1)}{\sin x - x}$$

$$9) \lim_{x \rightarrow 0} x \cdot \csc^2 \sqrt{2x}$$

$$10) \lim_{x \rightarrow 0} \frac{\sin x^2}{x \cdot \sin x}$$

$$(ans. : 1) \frac{5}{7}; 2) 0; 3) -2; 4) -\frac{1}{2}; 5) \frac{1}{4}; 6) \sqrt{2}; 7) -1; 8) 3; 9) \frac{1}{2}; 10) 1)$$

2- Find the velocity v if a particle's position at time t is $s = 180t - 16t^2$. When does the velocity vanish ? (ans.: 5.625)

3-If a ball is thrown straight up with a velocity of 32 ft./sec. , its high after t sec. is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ? (ans.: 1, 16)



3- A stone is thrown vertically upwards at 35 m./sec. . Its height is : $s = 35t - 4.9t^2$ in meter above the point of projection where t is time in second later :

a) What is the distance moved, and the average velocity during the 3rd sec. (from $t = 2$ to $t = 3$) ?

b) Find the average velocity for the intervals ($t = 2$ to $t = 2.5$) , ($t = 2$ to $t = 2.1$) .

(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91

4- A stone is thrown vertically upwards at 24.5 m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge t sec. later is $4.9t(5 - t)$ m. . If its velocity is v m./sec. , differentiate to find v in terms of t :

i) when is the stone at the ledge level ?

ii) find its height and velocity after 1 , 2 , 3 , and 6 sec.

iii) what meaning is attached to negative value of s ? a negative value of v ?

iv) when is the stone momentarily at rest ? what is the greatest height reached ?

v) find the total distance moved during the 3rd sec. .

(ans.: $v=24.5-9.8t$; i) 0,5; ii) 19.6,29.4,29.4,-29.4; 14.7,4.9, -4.9,-34.3;

iv) 2.5; 30.625; v) 2.45)

5-A particle moves in a straight line so that after t sec. it is s m. , from a fixed point O on the line , where $s = t^4 + 3t^2$. Find :

i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.

ii) The average acceleration between $t = 1$ and $t = 3$.

(ans.: i) 18, 54, 114; ii) 58)