Solution of differential equations by Laplace transforms

① اعادة كتابة المعادلة بدلالة تحويلات لابلاس
② ادارج المعطيات الاولية
③ اعادة تحويل المعادلة جبريا
④ نحل بالتحويل العكسي بطريقة الحل بالتجزئة

يجب ان نحصل على تحويلات المشتقات $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$

Transforms of derivatives :-

Let $f'(t)$ first derivatives.
let $f''(t)$ second derivatives.

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

Integration by parts :-

$$\int u\,dv = uv - \int v\,du$$

$$u = f(t) \quad , \quad dv = e^{-st}\, dt$$

$$\Rightarrow du = f'(t)\, dt \quad , \quad \int dv = \int e^{-st}\, dt$$

$$\boxed{du = f'(t)\, dt} \quad \therefore \quad \boxed{v = -\frac{1}{s} e^{-st}}$$

$$\int_0^\infty e^{-st} f'(t) \cdot dt = \left[-\frac{1}{s} e^{-st} \cdot f(t)\right]_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st} f'(t)\, dt \Big) * s$$

$$= e^{-st} \cdot f(t) \Big]_0^\infty - \int_0^\infty e^{-st} \cdot f'(t)\, dt$$

at $t = \infty$, $e^{-st} f(t) = 0$

Email (shahad.mahmood.mohammed@uomus.edu.iq)

$\therefore L[f'(t)] = 0 - e^{0} f(0) + s \int_0^{\infty} e^{-st} f(t) \cdot dt$

$L[f'(t)] = -f(0) + s \int_0^{\infty} e^{-st} f(t)\, dt$

$\therefore L[f'(t)] = sF(s) - f(0)$

For second derivatives.

$L[f''(t)] = L[(f')'(t)] = s L[f'(t)] - f'(0)$

$L[f'(t)] = sF(s) - f(0)$

$F(s) = L[f(t)]$

$L[f''(t)] = s[sF(s) - f(0)] - f'(0)$

$= s^2 F(s) - s f(0) - f'(0)$

$L(f(t)) = F(s)$

$L(f'(t)) = sF(s) - f(0)$

$L(f''(t)) = s^2 F(s) - s f(0) - f'(0)$

$L(f'''(t)) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$

Email (shahad.mahmood.mohammed@uomus.edu.iq)

Alternative :- البديل

يستخدم لترتيب الحل

let $x = f(t)$, and $t = 0$

$x = x_0 \Rightarrow f(0) = x_0$

$\frac{dx}{dt} = x_1 \Rightarrow f'(0) = x_1$

$\frac{d^2x}{dt^2} = x_2 \Rightarrow f''(0) = x_2$

$\therefore \frac{d^nx}{dt^n} = x_n \Rightarrow f^n(0) = x_n$

* denote the laplace transform of x by $\bar{x}$

$\bar{x} = L(x) = L[f(t)] = F(s)$.

* using the 'dot' for derivatives.

$L[x] = \bar{x}$

$L[\dot{x}] = s\bar{x} - x_0$

$L[\ddot{x}] = s^2\bar{x} - sx_0 - x_1$

$L[\dddot{x}] = s^3\bar{x} - s^2x_0 - sx_1 - x_2$

$L[\ddddot{x}] = s^4\bar{x} - s^3x_0 - s^2x_1 - sx_2 - x_3$

Solution of first-order D.E.

Ex: 1 / solve the eq. $\frac{dx}{dt} - 2x = 4$, at $t=0$, $x=1$

① نعيد كتابة المعادلة بتحويل لابلاس

$L[4] = \frac{4}{s}$

$L[f'(t)] = L[\dot{x}] = s\bar{x} - x_0$

② نرجع ترتيب المعادلة تعبير بصيغة لابلاس

$\therefore (s\bar{x} - x_0) - 2\bar{x} = \frac{4}{s}$

③ نعوض x عندما تكون $t=0 \Leftarrow x_0 = 1$

$s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$

$\Rightarrow \bar{x}(s-2) = \frac{4}{s} + 1$

$\Rightarrow \bar{x} = \frac{s+4}{s(s-2)}$

④ نستخدم طريقة التجزئة

$$\frac{s+4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Email (shahad.mahmood.mohammed@uomus.edu.iq)

$\therefore \; s+4 = A(s-2) + Bs$

① let $(s-2)=0 \Rightarrow s=2$

$\therefore \; 2+4 = A(0) + 2B$

$\Rightarrow \boxed{B = 3}$

② let $s=0 \Rightarrow 4 = A(0-2)+0$

$\Rightarrow \boxed{A = -2}$

⑤ نرجع نحوله عكسيا [inverse laplace]

$$\bar{X} = \frac{3}{s-2} - \frac{2}{s}$$

$$X = L^{-1}\left[\frac{s+4}{s(s-2)}\right] = L^{-1}\left[\frac{3}{s-2} - \frac{2}{3}\right]$$

$$X = 3e^{2t} - 2$$

EX 2/ $\frac{dx}{dt} + 2x = 10e^{3t}$ at $t=0$, $x=6$

① convert the equations to laplace transforms.

$$(s\bar{x} - x_0) + 2\bar{x} = \frac{10}{s-3}$$

② insert the initial conditions, $x_0 = 6$

$$s\bar{x} - 6 + 2\bar{x} = \frac{10}{s-3}$$

$$\Rightarrow \bar{x}(s+2) = \frac{10}{s-3} + 6$$

$$\Rightarrow \bar{x} = \frac{6s-8}{(s+2)(s-3)}$$

③ taking inverse trasforms

$$x = L^{-1}\left[\frac{6s-8}{(s+2)(s-8)}\right]$$

$$\frac{6s-8}{(s+2)(s-3} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$\Rightarrow A(s-3) + B(s+2) = 6s-8$$

① let $(s-3)=0 \Rightarrow s=3$

$$A(0) + B(3+2) = 10$$

Email (shahad.mahmood.mohammed@uomus.edu.iq)

$\Rightarrow B = 2$

let $(s+2)=0 \Rightarrow s=-2$

$A(-2-3)+B(0)=-20$

$\Rightarrow A=4$

$\therefore \bar{x} = \dfrac{6s-8}{(s+2)(s-3)} = \dfrac{4}{s+2} + \dfrac{2}{s-3}$

$\therefore x = L^{-1}\left[\dfrac{4}{s+2} + \dfrac{2}{s-3}\right] = 4e^{-2t} + 2e^{3t}$

Ex 3/ $\dfrac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$, at $t=0, x=0$

① $(s\bar{x} - x_0) - 4\bar{x} = \dfrac{2}{s-2} + \dfrac{1}{s-4}$

② $x_0 = 0 \quad \therefore s\bar{x} - 4\bar{x} = \dfrac{2}{s-2} + \dfrac{1}{s-4}$

③ $\therefore \bar{x} = \dfrac{2}{(s-2)(s-4)} + \dfrac{1}{(s-4)^2}$

④ $\dfrac{2}{(s-2)(s-4)} = \dfrac{A}{s-2} + \dfrac{B}{s-4}$

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(5) $\frac{2}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4}$

$\Rightarrow \quad 2 = A(s-4) + B(s-2)$

$*$ let $(s-4) = 0$

$\Rightarrow B = 1$

$*$ let $(s-2) = 0$

$\Rightarrow A = -1$

$\bar{x} = \frac{1}{s-4} - \frac{1}{s-2} + \frac{1}{(s-4)^2}$

$\rightarrow F(s+a) = e^{-at} * f(t)$

$\frac{1}{(s-4)^2} = F(s-4) = e^{-(-4)t} * t$

$\therefore \; x = e^{4t} - e^{2t} + te^{4t}$

Solution of second-order D.E

Ex 4/

solve :- $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$, at $t=0$, $x=5$, $\frac{dx}{dt}=7$

① $L[x] = \bar{x}$
$L[\dot{x}] = s\bar{x} - x_0$
$L[\ddot{x}] = s^2\bar{x} - sx_0 - x_1$

$$(s^2\bar{x} - sx_0 - x_1) - 3(s\bar{x} - x_0) + 2\bar{x} = \frac{2}{s-3}$$

② at $t=0$, $x_0 = 5$, $x_1 = 7$

$$\therefore (s^2\bar{x} - 5s - 7) - 3(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}$$

$$(s^2 - 3s + 2)\bar{x} - 5s + 8 = \frac{2}{s-3}$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s-3} + 5s - 8$$

$$= \frac{2 + 5s^2 - 23s + 24}{s-3}$$

$$\Rightarrow \bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

③ $\bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

$$5s^2 - 23s + 26 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

* let $(s-3)=0 \Rightarrow s=3$

$45-69+26 = A(0)+B(0)+C(2)(1)$
$\Rightarrow C=2$

* let $(s-2)=0 \Rightarrow s=2$

$20-46+26 = A(0)+B(1)(-1)+C(0)$
$\Rightarrow B=0$

* let $(s-1)=0 \Rightarrow s=1$

$5-23+26 = A(-1)(-2)+B(0)+C(0)$
$\Rightarrow A=4$

$$\bar{x} = \frac{4}{s-1} + 0 + \frac{1}{s-3}$$

$$x = 4e^{t} + e^{3t}$$

Ex 5/ $\frac{d^2x}{dt^2} - 4x = 24\cos 2t$ at $t=0$, $x=3$, $\frac{dx}{dt}=4$

① $(s^2\bar{x} - s x_0 - x_1) - 4\bar{x} = \frac{24s}{s^2+4}$

② at $t=0$, $x_0=3$, $x_1=4$

$$s^2\bar{x} - 3s - 4 - 4\bar{x} = 3s+4+\frac{24s}{s^2+4} = \frac{3s^3+4s^2+36s+16}{s^2+4}$$

$$\bar{x} = \frac{3s^3+4s^2+36s+16}{(s^2+4)(s-2)(s+2)}$$

$$\bar{x} = \frac{As+B}{s^2+4} + \frac{C}{s-C} + \frac{D}{s+2}$$

اذا كان المقام عامل تربيعي
غير قابل للتحليل مثل s^2+4
فالبسط يجب ان يكون من الدرجة
الاقل بدرجة واحدة من المقام
المقام : s^2+4
البسط : $AS+B$

$$3s^3+4s^2+36s+16 = (As+B)(s-2)(s+2) + C(s^2+4)(s+2) + D(s^2+4)(s-2) \quad \text{Eq I}$$

* let $s-2=0 \Rightarrow s=2$

$$11+16+72+16 \Rightarrow (As+B)(0) + C(2+4)(4) + D(0)$$

$$\Rightarrow C=4$$

* let $s+2=0 \Rightarrow s=-2$

$$-11+16-72+16 = (As+B)(0) + C(0) + D$$

$$\Rightarrow D=2$$

Sub D & C in Eq I

$$3s^3+4s^2+36s+16 = (As+B)(s-2)(s+2) + 4(s^2+4)(s+2) + 2(s^2+4)(s-2)$$

$$\Rightarrow As^3 + Bs^2 - 4As - 4B + (4s^3 + 8s^2 + 16s + 32) + (2s^3 - 4s^2 + 8s - 16) = 3s^3 + 4s^2 + 36s + 16$$

$$s^3(A + 4 + 2) + s^2(B + 8 - 4) + s(-4A + 16 + 8) + 4(-B + 8 - 4) = 3s^3 + 4s^2 + 36s + 16$$

نستخدم طريقة جمع المعاملات المتشابهة

at s^3: $A + 6 = 3 \Rightarrow A = -3$

at s^2: $B + 4 = 4 \Rightarrow B = 1$

$$\bar{X} = \frac{-3s + 1}{s^2 + 4} + \frac{4}{s - 2} + \frac{2}{s + 2}$$

$$\bar{X} = \frac{4}{s-2} + \frac{2}{s+2} - \frac{3s+1}{s^2+4}$$

$$\Rightarrow x = 4e^{2t} + 2e^{-2t} - 3\cos 2t$$

Ex 6/ Solve $\ddot{x} + 5\dot{x} + 6x = 4t$ at $t=0$, $x=0$, $\dot{x}=0$

$$(s^2\bar{x} - s x_0 - x_1) + 5(s\bar{x} - x_0) + 6\bar{x} = \frac{4}{s^2}$$

$x_0 = 0$, $x_1 = 1$

$$\therefore (s^2\bar{x} - 0 - 0) + 5(s\bar{x} - 0) + 6\bar{x} = \frac{4}{s^2}$$

$$\bar{x}(s^2 + 5 + 6) = \frac{4}{s^2}$$

$$\Rightarrow \bar{x} = \frac{4}{s^2(s^2+5+6)} = \frac{4}{s^2(s+2)(s+3)}$$

$$\Rightarrow \bar{x} = \frac{1}{s}\left[\frac{4}{s(s+2)(s+3)}\right] = \frac{1}{s}\left[\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}\right]$$

حل فقط داخل القوس يعني ناخذه بدون $\frac{1}{s}$

$$\frac{4}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$\Rightarrow 4 = A(s+2)(s+3) + B s(s+3) + C s(s+2)$$

by coverup rule :-

* at A let $s=0 \Rightarrow 4 = 6A + B(0) + C(0)$

$$\Rightarrow A = \frac{4}{6} = \frac{2}{3}$$

* at B let $s+2=0 \Rightarrow s=-2$

$$4 = A(0) + B-2(1) + C(0) \Rightarrow 4 = -2B$$

$$\Rightarrow B = -2$$

* at C let $S+3=0 \Rightarrow S=-3$

$$4 = A(0) + B(0) + C-3(-3+2) \Rightarrow 4=3C$$

$$\Rightarrow C = \frac{4}{3}$$

$$\therefore \bar{X} = \frac{1}{s}\left[\frac{2}{3}\frac{1}{s} + \frac{-2}{s+2} + \frac{4}{3}\frac{1}{s+3}\right]$$

$$\Rightarrow \bar{X} = \frac{2}{3}\cdot\frac{1}{s^2} - \frac{2}{s(s+2)} + \frac{4}{3}\frac{1}{s(s+3)} \longrightarrow Eq1$$

* $\frac{2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$ by partial fractions or coverup

* $\frac{4}{3}\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

* $\frac{2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$

$$\Rightarrow 2 = A(s+2) + Bs$$

at $s=0 \Rightarrow A=1$

at $s=-2 \Rightarrow 2 = 0-2B \Rightarrow B=-1$

$$\therefore \frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2} \longrightarrow Eq2$$

* $\frac{4}{3}\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

$$\Rightarrow \frac{4}{3} = A(s+3) + Bs$$

at $s = 0 \Rightarrow A = \frac{4}{9}$

at $s = -3 \Rightarrow \frac{4}{3} = 0 - 3B \Rightarrow B = -\frac{4}{9}$

$$\therefore \frac{4}{3}\frac{1}{s(s+3)} = \frac{4}{9}\cdot\frac{1}{s} - \frac{4}{9}\frac{1}{s+3} \quad \text{Eq3}$$

sub. Eq 2, 3 in 1

$$\bar{X} = \frac{2}{3}\cdot\frac{1}{s^2} - \left[\frac{1}{s} - \frac{1}{s+2}\right] + \left[\frac{4}{9}\cdot\frac{1}{s} - \frac{4}{9}\cdot\frac{1}{s+3}\right]$$

$$\bar{X} = \left(\frac{2}{3}\cdot\frac{1}{s^2}\right) - \frac{1}{s} + \frac{1}{s+2} + \left(\frac{4}{9}\cdot\frac{1}{s}\right) - \left(\frac{4}{9}\cdot\frac{1}{s+3}\right)$$

$$\bar{X} = \left(\frac{2}{3}\cdot\frac{1}{s^2}\right) - \left(\frac{5}{9}\cdot\frac{1}{s}\right) + \frac{1}{s+2} - \left(\frac{4}{9}\cdot\frac{1}{s+3}\right)$$

$$\therefore X = \frac{2}{3}t - \frac{5}{9} + e^{-2t} - \frac{4}{9}e^{-3t}$$

H.W 1 // solve $\ddot{x} - 2\dot{x} + 10x = e^{2t}$
at, $t = 0$, $x = 0$, $\dot{x} = 1$

find X ?

hint $[s^2 - 2s + 10] = (s-1)^2 + 9$