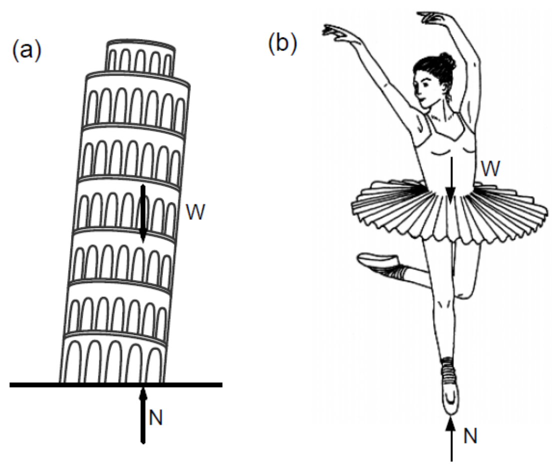
# Laws of Motion and Their Applications

## 1. Laws of Motion: A Historical Perspective

Newton's laws of motion form the foundation of classical mechanics, explaining how forces interact with objects in motion. These laws were first mathematically formulated by Sir Isaac Newton in 1687 in his *Philosophic Naturalis Principia Mathematica*.

**First Law (Law of Inertia):**

An object remains at rest or in uniform motion unless acted upon by an external force. This law is observed in various scenarios, such as a dog not moving unless a force acts on it (e.g., pulling the leash). Fig.1 a,b illustrates examples of static equilibrium and balance.




**Second Law (Law of Acceleration):**

According to the second law, an object will accelerate in the direction of the unbalanced force. It is expressed mathematically as:

$$\boldsymbol{F = ma}$$

where:

- $\boldsymbol{F}$ is the sum of all forces acting on the object.
- $\boldsymbol{m}$ is the mass of the object.
- $\boldsymbol{a}$ is the acceleration.

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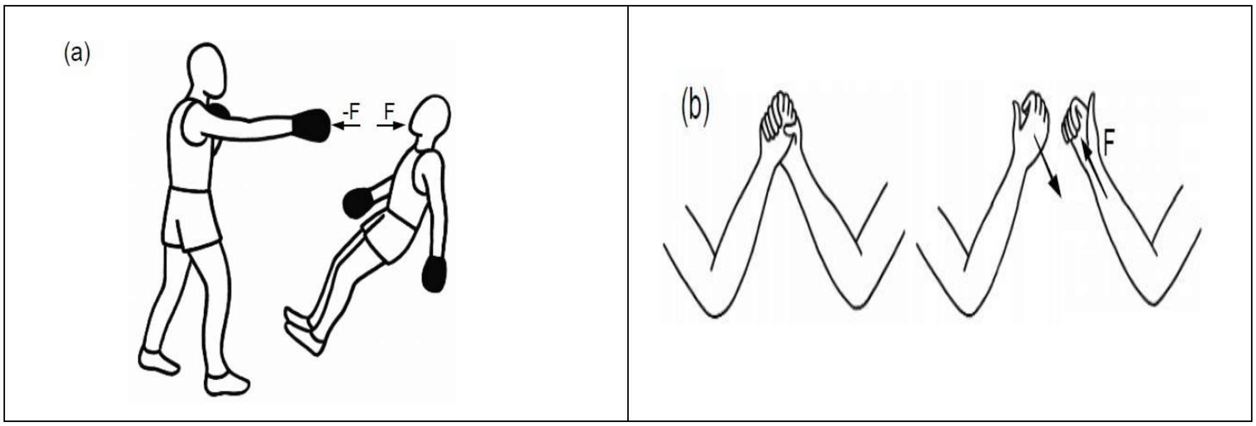
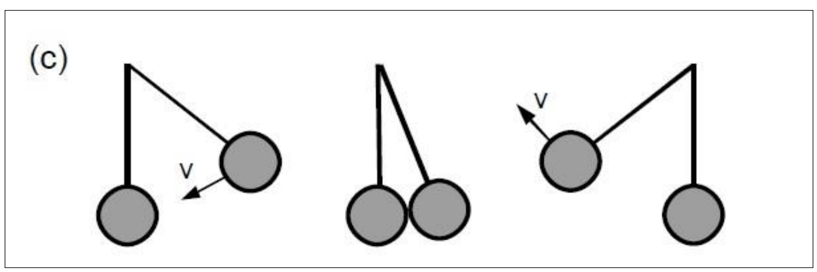
**Third Law (Action and Reaction):**

Newton's third law states that the force of reaction is equal in magnitude and opposite in direction to the force of action. This law is evident when a boxer punches an opponent both exert the same force on each other. Fig .2 a,b and c show examples of action-reaction pairs in different contexts.




## 2. Addition and Subtraction of Vectors

Vectors represent quantities that have both magnitude and direction. In mechanics, vectors are used to represent forces, velocity, and acceleration. To add vectors, we use the parallelogram law:

$$\vec{R} = \vec{a} + \vec{b}$$

where $\vec{a}$ and $\vec{b}$ are two vectors. The resultant vector $\vec{R}$ is obtained by connecting the tail of one vector to the head of the other. As shown in Fig. 3 a, b and c.

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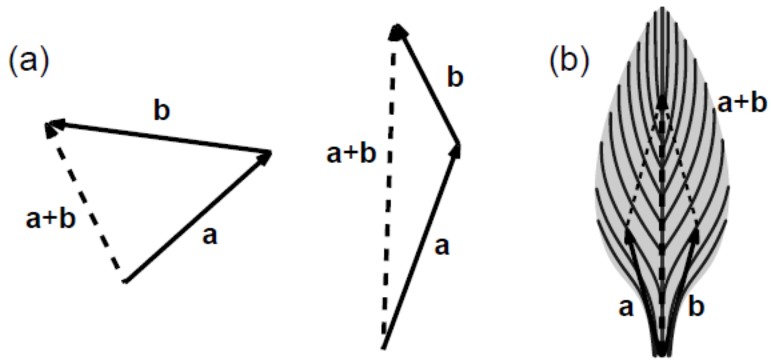



The following properties can also be derived:

- **Commutative Property**:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

This property implies that the order in which we add vectors does not affect the sum.

- **Associative Property**:

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$

This property indicates that we can group vectors in any way when adding them, and the result will be the same.

- **Subtraction of Vectors**:

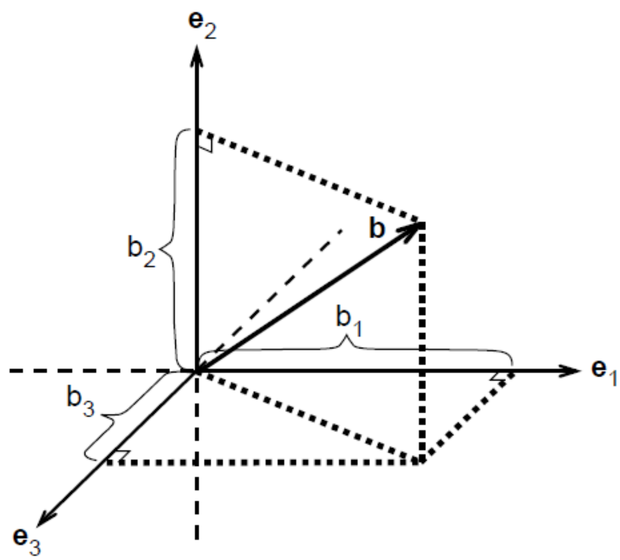
$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

The vector $-b$ has the same magnitude as $b$ but points in the opposite direction.

Let E ($x_1$, $x_2$, $x_3$) be a Cartesian coordinate system and let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ denote unit vectors along $x_1$, $x_2$, and $x_3$ directions as shown in Fig. 4. A unit vector is defined as a vector with a unit magnitude. Any vector **b** can be expressed in this coordinate system as the sum of three components:

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3$$

The components $b_1, b_2, b_3$ represent the projections of vector $b$ onto the $x_1, x_2, x_3$ axes, respectively as shown in Fig 4. To determine projection $b_1$, one draws perpendicular lines from each end of the vector **b** to the $x_1$ axis. The line segment that remains between the two intersections represents $b_1$. Projections $b_2$ and $b_3$ are determined similarly.




A projection is positive if it points along one of the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$; otherwise, it is negative.

The magnitude of a vector $b$ is given by:

$$|| b || = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

The unit vector along the direction of vector $b$ is:

$$e^b = \frac{b}{|| b ||}$$

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- **Equality and Operations on Vectors:**

Two vectors $a$and $b$are equal if and only if their components along the coordinate axes are equal:

$$a = b \text{ if and only if } a_1 = b_1; a_2 = b_2; a_3 = b_3$$

The addition of two vectors $a$and $b$can be done algebraically as:

$$a + b = (a_1 + b_1)e_1 + (a_2 + b_2)e_2 + (a_3 + b_3)e_3$$

- **Multiplication of Vectors by a Scalar:**

When a vector is multiplied by a positive scalar, its magnitude is multiplied by the scalar. If it is multiplied by a negative scalar, its direction is reversed, and its magnitude is multiplied by the absolute value of the scalar.

$$s(ta) = (st)a$$

$$(s + t)a = sa + ta$$

$$s(a + b) = sa + sb$$

**Example 1: Elbow Force During Baseball Pitching**

Baseball players, as shown in Fig.5 especially pitchers, are prone to overuse injuries. The forces applied by the ligaments and tendons on the elbow joint during pitching were measured in various directions. The magnitudes of these forces were found to be:

- $F_M = 428\ N, F_A = 101\ N, F_C = 253\ N.$

The unit vectors for the medial, anterior, and compression directions were given as:

- $e_M = 0.79e_1 + 0.17e_2 + 0.59e_3$
- $e_A = 0.21e_1 - 0.98e_2$
- $e_C = -0.58e_1 - 0.12e_2 - 0.81e_3$


*Solution*

By summing these forces, we compute the resultant force on the elbow joint:

$$F = F_M \cdot e_M + F_A \cdot e_A + F_C \cdot e_C$$

Substitute the values:

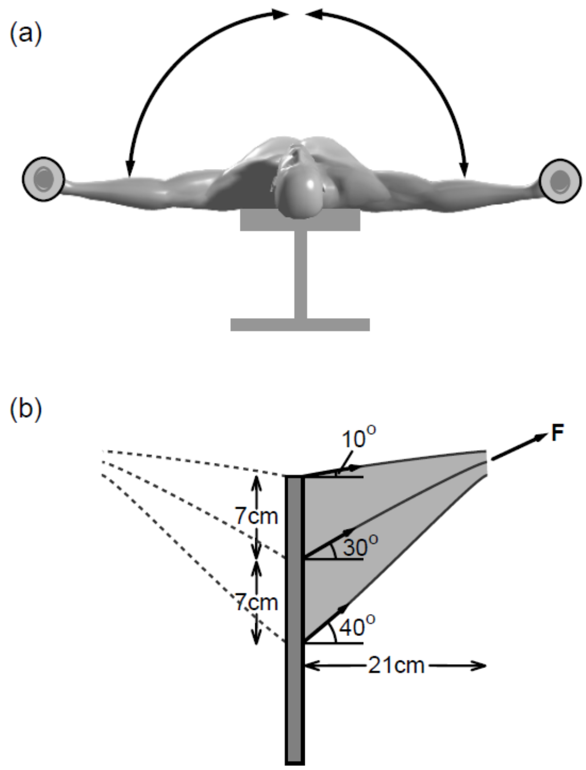
$$F = 428(0.79e_1 + 0.17e_2 + 0.59e_3) + 101(0.21e_1 - 0.98e_2) + 253(-0.58e_1 - 0.12e_2 - 0.81e_3)$$

$$F = (428 \cdot 0.79 + 101 \cdot 0.21 + 253 \cdot -0.58)e_1 + (428 \cdot 0.17 + 101 \cdot -0.98 + 253 \cdot -0.12)e_2 + (428 \cdot 0.59 + 101 \cdot 0 + 253 \cdot -0.81)e_3$$

$$F = 212.12e_1 - 56.97e_2 + 148.88e_3 \text{ N}$$

**Example.2: Resultant Force Exerted by the Pectoralis on the Upper Arm**

The pectoralis muscle is responsible for exerting force on the upper arm. To calculate the resultant force, we assume that the force produced by each set of muscle fibers is 75 N. The resultant pectoralis force $F$ as shown in Fig 6 is the sum of the forces produced by the three sets of fibers:




*Solution*

$$F = 75\,\mathrm{N}(\cos\ 40^\circ + \cos\ 30^\circ + \cos\ 10^\circ)e_1 + 75\,\mathrm{N}(\sin\ 40^\circ + \sin\ 30^\circ + \sin\ 10^\circ)e_2$$

We calculate the values:

$$F = 75\,\mathrm{N}(0.766 + 0.866 + 0.985)e_1 + 75\,\mathrm{N}(0.643 + 0.500 + 0.174)e_2$$
$$F = 75\,\mathrm{N}(2.617)e_1 + 75\,\mathrm{N}(1.317)e_2$$
$$F = 196.275e_1 + 98.775e_2\ \mathrm{N}$$

## 3. Time Derivatives of Vectors

The time derivative of a vector is crucial when studying motion. The derivative of position with respect to time gives velocity:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

and the derivative of velocity gives acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

## 4. Position, Velocity, and Acceleration

Position vectors describe the location of an object relative to a reference point. Velocity and acceleration describe how the position changes over time. The position vector $\vec{r}$can be written as:

$$\vec{r}(t) = (x_1 e_1 + x_2 e_2 + x_3 e_3)$$

where $x_1, x_2, x_3$are the coordinates as functions of time.

The velocity and acceleration are the time derivatives of position and velocity, respectively:

$$\vec{v} = \frac{d\vec{r}}{dt} = (\frac{dx_1}{dt} e_1 + \frac{dx_2}{dt} e_2 + \frac{dx_3}{dt} e_3)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (\frac{dv_1}{dt} e_1 + \frac{dv_2}{dt} e_2 + \frac{dv_3}{dt} e_3)$$

**Example.3: Particle Path, Velocity, and Acceleration**

Particle Path, Velocity, and Acceleration. The position vector connecting a fixed point ***O*** in the reference frame ***E*** to a moving point P in space is given by the expression:

$$\vec{r}_{P/O} = (1.67 + 3t^2)[\cos{(2t^2)} e_1 + \sin{(2t^2)} e_2]$$

Determine the velocity and acceleration of point ***P*** in reference frame ***E***. Solution: The unit vectors ***e₁, e₂*** are constants in ***E*** so their time derivative will be zero. Using differentiation by parts, we find

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(1.67 + 3t^2)[\cos{(2t^2)} e_1 + \sin{(2t^2)} e_2]$$

Using the product and chain rule:

$$\vec{v} = (6t[\cos(2t^2)e_1 + \sin(2t^2)e_2] + (1.67 + 3t^2)[-4t\sin(2t^2)e_1 + 4t\cos(2t^2)e_2])$$

Thus, the velocity vector is:

$$\vec{v} = (6t\cos(2t^2) - (1.67 + 3t^2)4t\sin(2t^2))e_1 + (6t\sin(2t^2) + (1.67 + 3t^2)4t\cos(2t^2))e_2$$

Now, differentiate again to find the acceleration vector:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

This involves applying the chain rule to each term in the velocity equation and solving for each component of acceleration. After differentiation, we obtain the full expression for acceleration as a function of time.

$$\vec{a} = \frac{d\vec{v}}{dt} = [6\cos(2t^2) - 24t^2\sin(2t^2) - (6.7 + 36t^2)\sin(2t^2) + (26.8t^2 + 48t^4)\cos(2t^2)]e_1 + [6\sin(2t^2) + 24t^2\cos(2t^2) + (6.7 + 36t^2)\cos(2t^2) - (26.8t^2 + 48t^4)\sin(2t^2)]e_2$$

## 4. Velocity and Acceleration in Polar Coordinates

In some cases, it might be easier to compute the velocity and acceleration in **polar coordinates** rather than **Cartesian coordinates**.

In polar coordinates, we define $e_r$ to be the unit vector in the direction of the position vector connecting the origin $O$ of the coordinate system to a moving point $P$. Consider, for example, the case of abduction of the arm (as shown in Fig. 7). The unit vector along the line of the arm $e_r$ is given by the equation:

$$e_r = \cos\theta\, e_1 + \sin\theta\, e_2$$

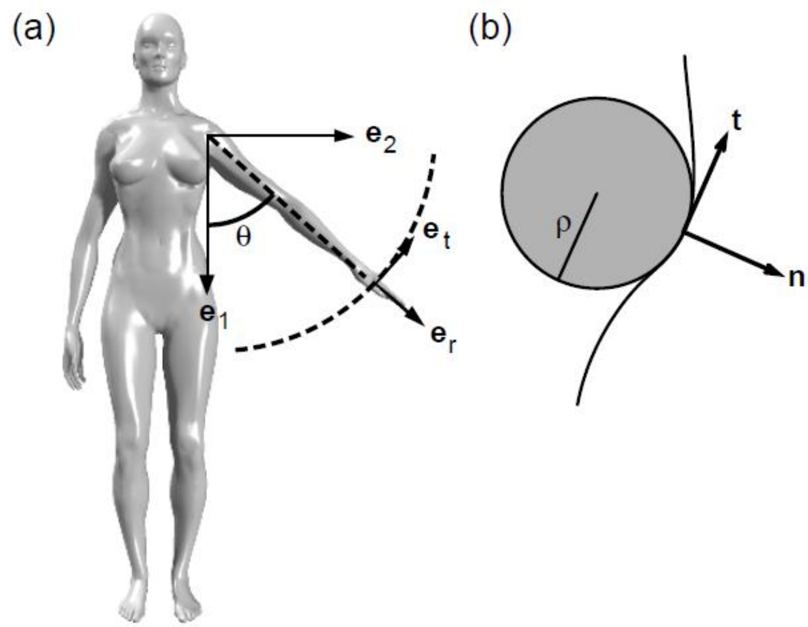



FIGURE.7 a,b. Polar and path coordinates. The unit vectors associated with polar coordinates are $e_r$ and $e_t$. The vector $e_r$ is in the radial direction pointing outward whereas $e_t$ is tangent to the circle and points in the direction of increasing $\theta$ (a). In the case of path coordinates, the unit vector n is normal and t is tangent to the trajectory (b). The symbol $\rho$ denotes the radius of curvature

Then, the position vector connecting the shoulder to the elbow can be written as:

$$\vec{r} = Le_r$$

where:

- $L$ denotes the length of the upper arm.

Taking the time derivative of the position vector, we determine the velocity of the elbow:

$$\vec{v} = L\frac{de_r}{dt}$$

Using the chain rule, we get:

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$$\vec{v} = L(\frac{d\theta}{dt})(-\sin\ \theta\, \hat{e_1} + \cos\ \theta\, \hat{e_2})$$

Simplifying:

$$\vec{v} = L(\frac{d\theta}{dt})\hat{e_t}$$

where:

- $\hat{e_t}$ is perpendicular to $\hat{e_r}$, as shown in **Fig.7**.

Next, we determine the acceleration by taking the time derivative of the velocity $\vec{v}$:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = L(\frac{d^2\theta}{dt^2})(-\sin\ \theta\, \hat{e_1} + \cos\ \theta\, \hat{e_2}) - L(\frac{d\theta}{dt})^2(\cos\ \theta\, \hat{e_1} + \sin\ \theta\, \hat{e_2})$$

Simplifying:

$$\vec{a} = L(\frac{d^2\theta}{dt^2})\hat{e_t} - L(\frac{d\theta}{dt})^2\hat{e_r}$$

Note that the **speed** of the particle $v$ is given by the expression:

$$v = L(\frac{d\theta}{dt})$$

**Example.4: Arm Movements in Aerobics**

An aerobic instructor abducts her arm from downward vertical position to horizontal position at shoulder length in 0.6 seconds (s), at constant rate (Fig.7a). Determine the velocity and acceleration of her elbow. Assume that the length of her upper arm is 0.38 m.

Solution: Since 0.6 s was required to traverse an angle of $\pi/2$ radians at constant rate

$$\frac{d\theta}{dt} = \frac{\pi}{1.2}$$

$$\frac{d^2\theta}{dt^2} = 0$$

where $L$is the length of the upper arm, the velocity and acceleration of her elbow are calculated as:

$$\vec{v} = L \cdot (\frac{d\theta}{dt})e_t = 0.99\ e_t$$

$$\vec{a} = L \cdot (\frac{d^2\theta}{dt^2})e_t - L(\frac{d\theta}{dt})^2 e_r = -2.6\ m/s^2 e_r$$

where $e_r$and $e_t$are the radial and tangential unit vectors.

## Problems

**Problem 1.** Consider a cable connecting two points A and B that are fixed in space (Fig. P.1). The cable is in tension so that it forms a straight line passing through A and B. Why is the tension in the cable uniform between these points? Take a small segment of the cable and draw all the forces that act on it. Use Newton's second law for this segment of the cable.

**Problem 2.** Why does the tension in an inextensible cable remain uniform when the cable goes over a frictionless pulley?

**Problem 3.** The coordinates of a point with respect to a Cartesian coordinate system E with unit vectors ($\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$) can be written as A ($x_1$, $x_2$, $x_3$) where A denotes the point under question. Let A (1, 5, −2) and B (3, −4, 6) be two points in space. Determine the position vector connecting A to B. Also determine the unit vector along the straight line from A to B.

*Answer:* $\mathbf{r}^{B/A} = 2\ \mathbf{e}_1 - 9\ \mathbf{e}_2 + 8\ \mathbf{e}_3$; $\mathbf{e} = 0.16\ \mathbf{e}_1 - 0.74\ \mathbf{e}_2 + 0.66\ \mathbf{e}_3$.

**Problem 4.** Determine the velocity and acceleration of a small child in a swing of length 3.5 m at a time when the swing is at 30° with the ver-

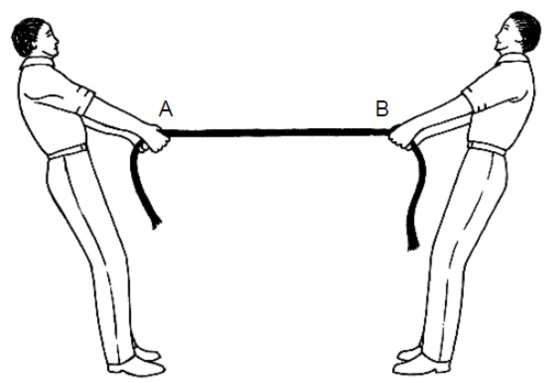


FIGURE P.1. Two men pulling on a rope.

tical axis. At that instant, the swing was moving up at a constant rate of $(\pi/8)$ rad/s.
*Answer:* $\mathbf{v} = 1.2\ \mathbf{e}_1 + 0.7\ \mathbf{e}_2$ (m/s); $\mathbf{a} = 0.27\mathbf{e}_1 + 0.47\mathbf{e}_2$ (m/s$^2$) where $\mathbf{e}_1$ and $\mathbf{e}_1$ are unit vectors in the plane of motion in the horizontal and vertical directions, respectively.

**Problem 5.** A volleyball, thrown from position $\mathbf{r} = 2\ \mathbf{e}_2$ at time $t = 0$, occupies the position $\mathbf{r} = 6\ \mathbf{e}_1 + 4\ \mathbf{e}_2$ at time $t = 2$ s. Determine the initial velocity of the ball. Determine the highest elevation (y) the ball reaches while airborne.
*Answer:* $\mathbf{v}_o = 3\ \mathbf{e}_1 + 10.8\ \mathbf{e}_2$ (m/s); $y = 7.9$ m.

**Problem 6.** In baseball, a pitcher standing 60 ft from the home plate needs about 0.4 s to raise his glove in defense of a hard line drive hit at him. In March 1996, a batter hit the pitcher in the face with a ball that traveled back in 0.29 s. Five of the pitcher's teeth were shattered, and he needed 65 stitches to mend his face. Determine the horizontal component of the velocity of the baseball as it hit the pitcher.
*Answer:* $v_1 = 207$ ft/s.