* Laplace transforms *

The Laplace transform an expression $f(t)$ is denoted by $\mathcal{L}[f(t)]$ [تحويل الدالة الزمنية إلى دالة مركبة (لابلاس)]

$$\mathcal{L}[f(t)] = \int_{t=0}^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

$F(s)$ = Laplace transformed

Ex1: Find Laplace transform of $f(t) = a$ (constant)

$$L(a) = \int_0^{\infty} a e^{-st}\, dt = -\frac{1}{s}\int_0^{\infty} a e^{-st} s\, dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{a}{s} e^{-st}\Big]_0^{\infty} = -\frac{a}{s}\left[\overset{0}{e^{-s\infty}} - \overset{1}{e^{-s0}}\right]$$

مشتقة

$$\therefore L(a) = \frac{a}{s} \quad (s > 0)$$

Ex 2: $f(t) = e^{at}$, multiply by e^{-st} and integrate between $t = 0$ ant $t = \infty$

$$\therefore L[e^{at}] = \int_0^{\infty} e^{at} e^{-st}\, dt = \int_0^{\infty} e^{-t(s-a)}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= -\frac{1}{s-a}[0-1] = \frac{1}{s-a}$$

$$\therefore L(e^{at}) = \frac{1}{s-a} \quad (s > a)$$

$\therefore$ at $L(4) = \frac{4}{s}$

$\therefore$ at $L(e^{4t}) = \frac{1}{s-4}$

Ex 3 :- $f(t) = \sin at$

$$L[\sin at] : \int_0^\infty \sin at \cdot e^{-st}\, dt$$

$$e^{i\theta} = \cos\theta + i\sin\theta \longrightarrow \text{Euler} \quad \therefore \sin\theta = e^{i\theta}$$

imaginary part

$$\therefore \sin(at) = \mathcal{I}(e^{iat})$$

الجزء التخيلي

$$\therefore L[\sin at] = L[\mathcal{I}(e^{iat})]$$

$$= \mathcal{I}\int_0^\infty e^{iat} \cdot e^{-st}\, dt = \mathcal{I}\int_0^\infty e^{-t(s-ia)} \cdot dt$$

$$= \mathcal{I}\left[\left(\frac{e^{-(s-ia)t}}{-(s-ia)}\right)_0^\infty\right] = \mathcal{I}\left[-\frac{1}{(s-ia)}[0-1]\right]$$

$$= \mathcal{I}\left[\frac{1}{s-ia}\right] \quad * \, s+ia$$

$$= \mathcal{I}\left[\frac{s+ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2} \rightarrow \text{imaginary part}$$

$$\therefore L[\sin at] = \frac{a}{s^2+a^2}$$

H.w 1 :- Determine $L[\cos at]$, since $\cos at$ is the real part of e^{iat} written $\mathcal{R}(e^{iat})$.

at $L[\sin 2t] = \dfrac{2}{s^2+4}$

at $L[e^{-3t}] = \dfrac{1}{s+3}$

Ex 4 :- $f(t) = \sinh at$, $f(t) = \cosh at$

exponential defintions of sinh at and cosh at

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$

$$\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$

$$\mathcal{L}[\sinh at] = \int_0^\infty \sinh at \, e^{-st} \, dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{at} - e^{-at}\right] e^{-st} \, dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} - e^{-(s+a)t}\right] dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}$$

$$\therefore \ L[\sinh at] = \frac{a}{s^2 - a^2}$$

$$L[\cosh at] = \frac{s}{s^2 - a^2}$$

H.W 2 :- prove it .

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* standard trasforms *

$f(t)$	$L[f(t)] = F(s)$	
a	$\dfrac{a}{s}$	
e^{at}	$\dfrac{1}{s-a}$	
$\sin at$	$\dfrac{a}{s^2+a^2}$	
$\cos at$	$\dfrac{s}{s^2+a^2}$	
$\sinh at$	$\dfrac{a}{s^2-a^2}$	
$\cosh at$	$\dfrac{s}{s^2-a^2}$	
t^n	$\dfrac{n!}{s^{n+1}}$	n : positive integer

laplace trasform is alinear trasform [Theorem 1]

① $L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$

② $L[\underbrace{K}_{\text{constant}} f(t)] = K\, L[f(t)]$

Ex5: a/ $L[2e^{-t} + t] = L[2e^{-t}] + L[t]$

$= 2L[e^{-t}] + L[t]$

$= \dfrac{2}{s+1} + \dfrac{1}{s^2} = \dfrac{2s^2+s+1}{s^2(s+1)}$

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b/ $L[2\sin 3t + \cos 3t] = 2L[\sin 3t] + L[\cos 3t]$

$$= 2 \cdot \frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9}$$

c/ $L[4e^{2t} + 3\cosh 4t] = 4L[e^{2t}] + 3L[\cosh 4t]$

$$= 4\frac{1}{s-2} + 3\frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16}$$

$$= \frac{7s^2-6s-64}{(s-2)(s^2-16)}$$

H-W 3 :

a/ $L[2\sin 3t + 4\sinh 3t] =$

b/ $L[t^3 + 2t^2 - 4t + 1] =$

First shifting Theorem s- Shifting [Theorem 2]

$L[e^{-at} f(t)] = F(s+a)$ Replacing s by s – a if f(t) multiply by e^{-at}

$$L[e^{-at} f(t)] = \int_0^\infty e^{-at} f(t) \cdot e^{-st} \cdot dt$$

$$= \int_0^\infty f(t) \cdot e^{-t(s+a)} \cdot dt$$

$\therefore = F(s+a)$ Because $L[f(t)] = \int_0^\infty f(t) e^{-st} \cdot dt = F(s)$

Ex 6:-

a/ $L[e^{-2t}\cosh 3t]$

$L[\cosh 3t] = \dfrac{s}{s^2-9}$

Replace s by $s+2$

$\therefore L[e^{-2t}\cosh 3t] = \dfrac{s+2}{(s+2)^2-9}$

b/ $L[2e^{3t}\sin 3t]$

$L[\sin 3t] = \dfrac{3}{s^2+9}$

$\therefore L[2e^{3t}\sin 3t] = 2\dfrac{3}{(s-3)^2+9} = \dfrac{6}{(s-3)^2+9}$

c/ $L[4te^{-t}]$

$L[4t] = \dfrac{4}{s^2}$

$\therefore L[4te^{-t}] = \dfrac{4}{(s+1)^2}$

d/ $L[e^{2t}\cos t]$

$L[\cos t] = \dfrac{s}{s+1}$

$\therefore L[e^{2t}\cos t] = \dfrac{s-2}{(s-2)^2+1} = \dfrac{s-2}{s^2-4s+5}$

Multiplying by t and t^n [Theorm 3]

If $L[f(t)] = F(s)$ then $L[tf(t)] = -F'(s)$

$$L[tf(t)] = \int_0^\infty t f(t) e^{-st} dt \qquad \frac{d e^{-st}}{ds} = -t e^{-st}$$

$$\therefore = \int_0^\infty f(t)\left(-\frac{d e^{-st}}{ds}\right)dt = -\frac{d}{ds}\int_0^\infty f(t) e^{-st} dt$$

$$= -F'(s)$$

$$\therefore L[tf(t)] = -F'(s)$$

Ex7:

a/ $L[t\sin 2t] = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$

b/ $L[t\cosh 3t] = -\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = \frac{(s^2-9) - s(2s)}{(s^2-9)^2}$

$$= -\frac{s^2-9-2s^2}{(s^2-9)^2} = \frac{s^2+9}{(s^2-9)^2}$$

c/ $L[t^2\cosh 3t] = L[t(t\cosh 3t)]$

$$= -\frac{d}{ds}\left[\frac{s^2+9}{(s^2-9)^2}\right]$$

$$= -\frac{2s(s^2-9)^2 - 4s(s^2-9)(s^2+9)}{(s^2-9)^4} = \frac{-2s(s^2-9)\left[(s^2-9) - 2(s^2+9)\right]}{(s^2-9)^4}$$

$$= \frac{-2s\cancel{(s^2-9)}(-s^2-27)}{(s^2-9)^{\cancel{4}\,3}} = \frac{2s(s^2+27)}{(S^2-9)^3}$$

H.w 4: find $L[t^2 \sin 4t]$

Dividing by t [Theorm 4]

if $L[f(t)] = F(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\,d\sigma$

ملاحظة: σ متغير صوري في التكامل لا يفرق النتيجة يستخدم فقط في داخل التكامل

$$\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \left[\int_0^\infty f(t)e^{-\sigma t}\,dt\right] d\sigma$$

$$= \int_s^\infty \int_0^\infty f(t)e^{-\sigma t}\,d\sigma\,dt$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-\sigma t}\,d\sigma\right] dt$$

$$= \int_0^\infty f(t)\,\frac{e^{-st}}{t}\,dt$$

$$\therefore = L\left[\frac{f(t)}{t}\right]$$

notes:- This rule is applicable only if $\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exist

In indeterminate cases we use L'Hopitals rule $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ to find out

$\therefore$ L'Hopitals rule:- $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Ex 8:

a/ $L\left[\frac{\sin at}{t}\right]$

$\lim_{t\to 0}\left[\frac{\sin at}{t}\right] = \frac{0}{0}$ ∴ By L'Hopitals rule

$\lim_{t\to 0}\left[\frac{a\cos at}{1}\right] = a$ ∴ the limit Exists

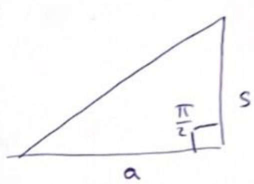
$\therefore L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{\sigma^2+a^2}\,d\sigma$ $\left[L(\sin at) = \frac{a}{s^2+a^2}\right]$

$= \left[\tan^{-1}\frac{\sigma}{a}\right]_s^\infty$ $\left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

at $\sigma = \infty$ $\tan^{-1}\left(\frac{\sigma}{a}\right) = \frac{\pi}{2}$

$\therefore L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$

$\tan^{-1}\frac{a}{s} + \tan^{-1}\frac{s}{a} = \frac{\pi}{2}$



$\therefore L\left[\frac{\sin at}{t}\right] = \tan^{-1}\frac{a}{s}$

tan	$\tan^{-1}$	θ
0	0	0°
1	45	45°
∞	90	90°

b/ $L\left[\frac{1-\cos 2t}{t}\right]$

$$\lim_{t\to 0}\left[\frac{1-\cos 2t}{t}\right] = \frac{0}{0}$$

By L'Hopitals $\lim_{t\to 0}\left[\frac{2\sin 2t}{t}\right] = \frac{0}{1} = 0$ ∴ limit exists

$$L[1-\cos 2t] = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left[\frac{1-\cos 2t}{t}\right] = \int_{\delta=s}^{\infty}\left[\frac{1}{\delta} - \frac{\delta}{\delta^2+4}\right]d\delta$$

$$= \left[\ln\delta - \frac{1}{2}\ln(\delta^2+4)\right]_s^{\infty} = \frac{1}{2}\left[\ln\left(\frac{\delta^2}{\delta^2+4}\right)\right]_s^{\infty}$$

$\ln\infty = 0$

$$\therefore = -\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right) = \ln\left(\frac{s^2}{s^2+4}\right)^{-\frac{1}{2}}$$

$$= \ln\sqrt{\frac{s^2+4}{s^2}}$$

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