Al-Mustaqbal University / College of Engineering
Department (Prosthetic and orthotic Engineering)
Class (1st)
Subject (Mathematics I) / Code (UOMU013013)
Lecturer (Assist Prof Dr. Firas Thair Al-Maliky)1st term – Lecture No. 2&3,
Lecture Name: Limits and Continuity

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Limits and continuity :

Limits : The limit of $F(t)$ as t approaches C is the number L if :

Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about C such that for all t, $0 < |t - C| < \delta$ implies $|F(t) - L| < \varepsilon$ and we can write it as :

$$\lim_{t \to C} F(t) = L$$

The limit of a function $F(t)$ as $t \to C$ never depend on what happens when $t = C$.

Right hand limit : $\quad \lim_{t \to C^+} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the right equals L if :
Given any $\varepsilon > 0$ (radius about L) there exists a $\delta > 0$ (radius to the right of C) such that for all t :

$$C < t < C + \delta \Rightarrow |F(t) - L| < \varepsilon$$

Left hand limit : $\quad \lim_{t \to C^-} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the left equal L if :
Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all t :

$$C - \delta < t < C \Rightarrow |F(t) - L| < \varepsilon$$

The limit combinations theorems :

1) $\lim[F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$

2) $\lim[F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$

3) $\lim \dfrac{F_1(t)}{F_2(t)} = \dfrac{\lim F_1(t)}{\lim F_2(t)} \quad$ where $\quad \lim F_2(t) \neq 0$

4) $\lim[k * F_1(t)] = k * \lim F_1(t) \quad \forall k$

5) $\lim_{\theta \to 0} \dfrac{Sin\,\theta}{\theta} = 1$

provided that θ is measured in radius

The limits (in $1 - 4$) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

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The continuity test **: The function** $y = f(x)$ **is continuous at** $x = C$ **if and only if all three of the following statements are true :**

1) $f(C)$ **exist (** C **is in the domain of** f **).**
2) $\lim\limits_{x \to C} f(x)$ **exists (** f **has a limit as** $x \to C$ **).**
3) $\lim\limits_{x \to C} f(x) = f(C)$ **(the limit equals the function value) .**

Thm.-2 **:** *The limit combination theorem for continuous function :*
If the function f **and** g **are continuous at** $x = C$ **, then all of the following combinations are continuous at** $x = C$ **:**

$$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g \quad \forall k \quad 4)\ g_{o}f\ ,\ f_{o}g \quad 5)\ f/g$$

$$provided\quad g(C) \neq 0$$

Thm.-3 **: A function is continuous at every point at which it has a derivative . That is , if** $y = f(x)$ **has a derivative** $f'(C)$ **at** $x = C$ **, then** f **is continuous at** $x = C$ **.**

Thm. -1 : The sandwich theorem **: Suppose that** $f(t) \leq g(t) \leq h(t)$ **for all** $t \neq C$ **in some interval about** C **and that** $f(t)$ **and** $h(t)$ **approaches the same limit** L **as** $t \to C$ **, then :**

$$\lim\limits_{t \to C} g(t) = L$$

Infinity as a limit **:**

1. **The limit of the function** $f(x)$ **as** x **approaches infinity is the number** L **:**
$\lim\limits_{x \to \infty} f(x) = L$ **. If , given any** $\varepsilon > 0$ **there exists a number** M **such that for all** x **:** $M < x \quad \Rightarrow \quad |f(x) - L| < \varepsilon$ **.**

2. **The limit of** $f(x)$ **as** x **approaches negative infinity is the number** L **:**
$\lim\limits_{x \to -\infty} f(x) = L$ **. If , given any** $\varepsilon > 0$ **there exists a number** N **such that for all** x **:** $x < N \quad \Rightarrow \quad |f(x) - L| < \varepsilon$ **.**

The following facts are some times abbreviated by saying :

a) **As** x **approaches** 0 **from the right ,** $1/x$ **tends to** ∞ **.**
b) **As** x **approaches** 0 **from the left ,** $1/x$ **tends to** $-\infty$ **.**
c) **As** x **tends to** ∞ **,** $1/x$ **approaches** 0 **.**
d) **As** x **tends to** $-\infty$ **,** $1/x$ **approaches** 0 **.**

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Continuity :

Continuity at an interior point : A function $y = f(x)$ is continuous at an interior point C of its domain if : $\lim\limits_{x \to C} f(x) = f(C)$.

Continuity at an endpoint : A function $y = f(x)$ is continuous at a left endpoint a of its domain if : $\lim\limits_{x \to a^+} f(x) = f(a)$.

A function $y = f(x)$ is continuous at a right endpoint b of its domain if: $\lim\limits_{t \to b^-} f(x) = f(b)$.

Continuous function : A function is continuous if it is continuous at each point of its domain .

Discontinuity at a point : If a function f is not continuous at a point C, we say that f is discontinuous at C, and call C a point of discontinuity of f .

Example: Find

1) $\lim\limits_{x \to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$, 2) $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x^4 - a^4}$

3) $\lim\limits_{x \to 0} \dfrac{Sin5x}{Sin3x}$, 4) $\lim\limits_{y \to 0} \dfrac{tan2y}{3y}$

5) $\lim\limits_{x \to 0} \dfrac{Sin2x}{2x^2 + x}$, 6) $\lim\limits_{x \to \infty} \left(1 + Cos\dfrac{1}{x} \right)$

7) $\lim\limits_{x \to \infty} \dfrac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5}$, 8) $\lim\limits_{y \to \infty} \dfrac{3y + 7}{y^2 - 2}$

9) $\lim\limits_{x \to \infty} \dfrac{x^3 - 1}{2x^2 - 7x + 5}$, 10) $\lim\limits_{x \to -1^-} \dfrac{1}{x + 1}$

11) $\lim\limits_{x \to 0} Cos\left(1 - \dfrac{Sinx}{x} \right)$, 12) $\lim\limits_{x \to 0} Sin\left(\dfrac{\pi}{2} Cos(tan x) \right)$

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_S01.-_

1) $\lim\limits_{x\to0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2} = \lim\limits_{x\to0} \dfrac{5x + 8}{3x^2 - 16} = \dfrac{0 + 8}{0 - 16} = -\dfrac{1}{2}$

2) $\lim\limits_{x\to a} \dfrac{x^3 - a^3}{x^4 - a^4} = \lim\limits_{x\to a} \dfrac{(x-a)(x^2 + ax + a^2)}{(x-a)(x+a)(x^2 + a^2)} = \dfrac{a^2 + a^2 + a^2}{(a+a)(a^2 + a^2)} = \dfrac{3}{4a}$

3) $\lim\limits_{x\to0} \dfrac{5\dfrac{Sin5x}{5x}}{3\dfrac{Sin3x}{3x}} = \dfrac{5}{3} \cdot \dfrac{\lim\limits_{5x\to0} \dfrac{Sin5x}{5x}}{\lim\limits_{3x\to0} \dfrac{Sin3x}{3x}} = \dfrac{5}{3}$

4) $\lim\limits_{y\to0} \dfrac{tan\,2y}{3y} = \dfrac{2}{3} \cdot \lim\limits_{2y\to0} \dfrac{Sin2y}{2y} \cdot \lim\limits_{y\to0} \dfrac{1}{Cos2y} = \dfrac{2}{3}$

5) $\lim\limits_{x\to0} \dfrac{Sin2x}{2x^2 + x} = 2 \lim\limits_{2x\to0} \dfrac{Sin2x}{2x} \cdot \lim\limits_{x\to0} \dfrac{1}{2x + 1} = 2$

6) $\lim\limits_{x\to\infty} \left(1 + Cos\dfrac{1}{x}\right) = 1 + Cos0 = 2$

7) $\lim\limits_{x\to\infty} \dfrac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5} = \lim\limits_{x\to\infty} \dfrac{3 + \dfrac{5}{x} - \dfrac{7}{x^3}}{10 - \dfrac{11}{x^2} + \dfrac{5}{x^3}} = \dfrac{3}{10}$

8) $\lim\limits_{y\to\infty} \dfrac{3y + 7}{y^2 - 2} = \lim\limits_{y\to\infty} \dfrac{\dfrac{3}{y} + \dfrac{7}{y^2}}{1 - \dfrac{2}{y^2}} = \dfrac{0}{1} = 0$

9) $\lim\limits_{x\to\infty} \dfrac{x^3 - 1}{2x^2 - 7x + 5} = \lim\limits_{x\to\infty} \dfrac{1 - \dfrac{1}{x^3}}{\dfrac{2}{x} - \dfrac{7}{x^2} + \dfrac{5}{x^3}} = \dfrac{1}{0} = \infty$

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10) $\lim\limits_{x\to-1^-} \dfrac{1}{x+1} = \dfrac{1}{-1+1} = -\infty$

11) $\lim\limits_{x\to0} Cos\left(1-\dfrac{Sinx}{x}\right) = Cos\left(1 - \lim\limits_{x\to0}\dfrac{Sinx}{x}\right) = Cos0 = 1$

12) $\lim\limits_{x\to0} Sin\left(\dfrac{\pi}{2}Cos(\tan x)\right) = Sin\left(\dfrac{\pi}{2}Cos(\tan 0)\right) = Sin\left(\dfrac{\pi}{2}Cos0\right) = Sin\dfrac{\pi}{2} = 1$

_EX-_ Test continuity for the following function :

$$f(x) = \begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0\le x<1 \\ 1 & x=1 \\ -2x+4 & 1<x\le2 \\ 0 & 2<x\le3 \end{cases}$$

Sol.- We test the continuity at midpoints $x = 0, 1, 2$ and endpoints $x = -1, 3$.

At $x=0 \Rightarrow$ $\qquad f(0) = 2*0 = 0$

$$\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0}(x^2-1) = -1$$
$$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0} 2x = 0 \ne \lim\limits_{x\to0^-} f(x)$$

Since $\lim\limits_{x\to0} f(x)$ doesn't exist

Hence the function discontinuous at $x=0$

At $x=1 \Rightarrow$ $\quad f(1) = 1$

$$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1} 2x = 2$$
$$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1}(-2x+4) = 2 = \lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1} f(x)$$

Since $\lim\limits_{x\to1} f(x) \ne f(1)$

Hence the function is discontinuous at $x=1$

At $x=2 \Rightarrow$ $\quad f(2) = -2*2+4 = 0$

$$\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2}(-2x+4) = 0$$
$$\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2} 0 = 0 = \lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2} f(x)$$

Since $\lim\limits_{x\to2} f(x) = f(2) = 0$

Hence the function is continuous at $x=2$

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$At \quad x = -1 \Rightarrow \quad f(-1) = (-1)^2 - 1 = 0$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1}(x^2 - 1) = 0 = f(-1)$$

$Hence \quad the \ function \ is \ continuous \ at \ x = -1$

$At \quad x = 3 \Rightarrow \quad f(3) = 0$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} 0 = 0 = f(3)$$

$Hence \quad the \ function \ is \ continuous \ at \ x = 3$

EX- What value should be assigned to a to make the function :

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases} \quad \text{continuous at } x = 3 \ ?$$

Sol. –

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) \Rightarrow \lim_{x \to 3}(x^2 - 1) = \lim_{x \to 3} 2ax \Rightarrow 8 = 6a \Rightarrow a = \frac{4}{3}$$

H.W

1. Discuss the continuity of :

$$f(x) = \begin{cases} x + \dfrac{1}{x} & for & x < 0 \\ -x^3 & for & 0 \leq x < 1 \\ -1 & for & 1 \leq x < 2 \\ 1 & for & x = 2 \\ 0 & for & x > 2 \end{cases}$$

(ans.: discontinuous at x=0,2 ; continuous at x=1)

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2. Evaluate the following limits :

a) $\lim\limits_{x \to \infty} \dfrac{x + Sinx}{2x + 5}$

b) $\lim\limits_{x \to \infty} \dfrac{1 + Sinx}{x}$

c) $\lim\limits_{x \to 0} \dfrac{x}{tan\,3x}$

d) $\lim\limits_{x \to \infty} \dfrac{x.Sinx}{(x + Sinx)^2}$

e) $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x}$

f) $\lim\limits_{x \to 1} \dfrac{\sqrt{x + 1} - \sqrt{2x}}{x^2 - x}$

g) $\lim\limits_{n \to \infty} (\sqrt{n^2 + 1} - n)$

(ans.:a)1/2, b)0, c)1/3, d)0, e)1/2, f)-1/2√2, g)0)

Reference

1-Nacy and Haabeeb, "Lectures Mathematics for 1st Class student, Technology University.

2- George B. Thomas Jr., "CALCULUS", 14th Ed