Euler_Cauchy eqs. & Applications

في البداية يجب ان نعرف صيغة المعادلة و درجتها

① order of D.E (High deirevative) اعلى مشتقة

Ex :- $2\frac{dy}{dx} - 3y - x^2$ (first order)

$\frac{d^2y}{dx^2} + 3y\left(\frac{dy}{dx}\right)^3 = x^5$ (second order)

② $y = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$, $y''' = \frac{d^3y}{dx^3}$

– linear eq :- $y'' + p(x)y' + q(x)y = r(x)$

– homogeneous eq :- $y'' + p(x)y' + q(x)y = 0$

If $r(x) \neq 0$ is nonhomogeneous linear D.E

Ex :- $y'' + 25y = e^{-x}\cos x$ (nonhomo. linear D.E)

$xy'' + y' + xy = 0$

or $y'' + \frac{1}{x}y' + y = 0$ [homo. Linear. D.E]

$y''y + y'^2 = 0$ [non linear D.E]

$p(x), q(x)$:- coefficients.

Euler_ Cauchy eq :-

حل معادلات تفاضلية خطية ذات معامل تعتمد على X.

General form :- $x^2y'' + axy' + by = 0$

General Solution :-

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

نعوضها في المعادلة العامة

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

$$x^{2+m-2}(m^2 - m) + amx^{1+m-1} + bx^m = 0$$

$$x^m(m^2-m) + amx^m + bx^m = 0 \quad \div m$$

$$m^2 - m + am + b = 0$$

معادلة المعامل / المعادلة المميزة

$$m^2 + (a-1)m + b = 0$$

$$m = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

$$\therefore m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}$$

$$m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$$

The root

#case 1 Real different root

If m_1 and m_2 (جذور حقيقية) give two real solutions

$$y_1(x) = x^{m_1} \quad , \quad y_2(x) = x^{m_2}$$

$$\therefore \quad y = c_1 x^{m_1} + c_2 x^{m_2}$$

$$m^2 + (a-1)m + b = 0$$

EX1// The Euler- cauchy equation $x^2y'' + 1.5xy' - 0.5y = 0$
$m^2 + 0.5m - 0.5 = 0$ The roots are 0.5 and -1
$y_1 = x^{0.5}$
$y_2 = 1/x$

Solution// $m_1 = 0.5$, $m_2 = -1$] جذور حقيقية ومختلفة

$$y_1 = x^{0.5} \quad , \quad y_2 = \frac{1}{x} = x^{-1}$$

$$\therefore \quad y = c_1 x^{0.5} + c_2 x^{-1}$$

$$\Rightarrow \quad y = c_1 \sqrt{x} + \frac{c_2}{x}$$

Case 2// A real double root

if $m_1 = m_2$ if $b = \frac{1}{4}(a-1)^2$

جذور متساوية

$\therefore m = \frac{1}{2}(a-1)$

$\therefore y_1 = x^m = x^{(1-a)/2}$

$y_1 = x^m$

$y_2 = x^m \ln x$

$\therefore y = [c_1 + c_2 \ln x]\, x^m$

$m^2 + (a-1)m + b = 0$

EX2// The Euler–Cauchy Equation $x^2y'' - 5xy' + 9y = 0$

solution It has the double root

$a = -5 \quad , b = 6$

$m^2 + (a-1)m + b = 0 \Rightarrow m^2 - 6m + 9 = 0$

$m = \frac{1}{2}(1+5) = 3$

$\therefore y = [c_1 + c_2 \ln x]\, x^3$

Case 3 Complex conjugate roots

اذا كانت الجذور معقدة

$x^{i\beta} = e^{i\beta \ln x}$

$e^{i\theta} = \cos\theta + i\sin\theta$ صيغة اويلر

$x^{i\beta} = \cos(\beta \ln x) + i\sin(\beta \ln x)$

$\therefore \; m = \alpha \pm i\beta$

$y = (c_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x))\, x^{\alpha}$

$m^2 + (a-1)m + b = 0$

EX3// The Euler Cauchy eq. $x^2y'' + 0.6xy' + 16.04y = 0$
The root are complex conjugate
$m_1 = 0.2 + 4i$ and $m_2 = 0.2 - 4i$, $i = \sqrt{-1}$

solution//

$m^2 + (0.6-1)m + 16.04 = 0$

$m^2 - 0.4m + 16.04 = 0$

$x^{m_1} = x^{0.2+4i} = x^{0.2} \cdot x^{4i} = x^{0.2}(e^{i4\ln x}) = x^{0.2} e^{(4\ln x)i}$

$x^{m_2} = x^{0.2-4i} = x^{0.2} \cdot x^{-4i} = x^{0.2}(e^{-i4\ln x}) = x^{0.2} e^{(-4\ln x)i}$

$\Rightarrow x^{m_1} = x^{0.2}[\cos(4\ln x) + i\sin(4\ln x)]$

$x^{m_2} = x^{0.2}[\cos(4\ln x) - i\sin(4\ln x)]$

$y = x^{0.2}[c_1\cos(4\ln x) + C_2\sin(4\ln x)]$

Applications

Boundry value problem Electric potential Field Between Two Concentric spheres :-

المجال الكهربائي في الاجسام الكروية

Ex:- Find the electrostatic potential $v = v(r)$ between two concentric sphere of radii $r_1 = 5$cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$V and $v_2 = 0$, respectively.

physical Information $v(r)$ is a solution of the Euler-Cauchy eq.

$$r v'' + 2v' = 0 \quad , \text{where } v' = dv/dr$$

solution :- $a = 2$, $b = 0$

$$m^2 + (2-1)m + 0 = 0$$

$$m^2 + m = 0 \Rightarrow m(m+1) = 0$$

$$\therefore m_1 = 0 \quad , m_2 = -1$$

$$v_1(r) = r^0 = 1 \quad , v_2(r) = r^{-1} = 1/r$$

$$\therefore v(r) = C_1 + \frac{C_2}{r}$$

at $r = 5$ $\quad v(5) = C_1 + \frac{C_2}{5} = 110 \quad \text{—— (1)}$

at $r = 10$ $\quad v(10) = C_1 + \frac{C_2}{10} = 0 \Rightarrow C_1 = -\frac{C_2}{10} \quad \text{—— (2)}$

sub (1) in (2)

$$\frac{C_2}{5} - \frac{C_2}{10} = 110 \Rightarrow C_2\left(\frac{1}{5} - \frac{1}{10}\right) = 110$$

$$\Rightarrow C_2 \frac{1}{10} = 110 \Rightarrow C_2 = 1100$$

sub C_2 in (2)

$C_1 + \frac{1100}{5} = 0 \Rightarrow \boxed{C_1 = -110}$

$\therefore v(r) = -110 + \frac{1100}{r} \, V$