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Lecturer (Assist Prof Dr. Firas Thair Al-Maliky)1st term
Lecture No 1,2&3. Lecture Name: Integration

Integration

*Integration is a fundamental concept in Calculus that is used to determine the accumulation of quantities and the area under a curve. It is defined as the mathematical process of finding the **integral of a function**, which represents the reverse operation of differentiation.*

There are two main types of integration:

1. ***Indefinite Integral** – used to find the general antiderivative of a function.*
2. ***Definite Integral** – used to calculate the exact accumulated value of a function between two limits.*

Indefinite integrals

The set of all anti derivatives of a function is called indefinite integral of the function. Assume u and v denote differentiable functions of x, and a, n, and c are constants, then the integration formulas are:-

1) $\int du = u(x) + c$

2) $\int a \cdot u(x)dx = a\int u(x)dx$

3) $\int \left(u(x) \mp v(x)\right)dx = \int u(x)\,dx \mp \int v(x)\,dx$

4) $\int u^n du = \dfrac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1}du = \int \dfrac{1}{u}du = \ln u + c$

5) $\int a^u du = \dfrac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$

EXAMPLE 1 Using the Power Rule

$$\int \sqrt{1+y^2} \cdot 2y\,dy = \int \sqrt{u} \cdot \left(\frac{du}{dy}\right) dy \qquad \text{Let } u = 1 + y^2,\ du/dy = 2y$$

$$= \int u^{1/2}\,du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C \qquad \text{Replace } u \text{ by } 1 + y^2.$$

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EXAMPLE 2 Adjusting the Integrand by a Constant

$$\int \sqrt{4t-1}\,dt = \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt$$

$$= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt \qquad \text{Let } u = 4t-1,\ du/dt = 4.$$

$$= \frac{1}{4}\int u^{1/2}\,du \qquad \text{With the 1/4 out front, the integral is now in standard form.}$$

$$= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{1}{6}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{1}{6}(4t-1)^{3/2} + C \qquad \text{Replace } u \text{ by } 4t-1.$$

EXAMPLE-3 – Evaluate the following integrals:

1) $\int 3x^2\,dx$

2) $\int\left(\frac{1}{x^2}+x\right)dx$

3) $\int x\sqrt{x^2+1}\,dx$

4) $\int\left(2t+t^{-1}\right)^2 dt$

5) $\int\sqrt{(z^2-z^{-2})^2+4}\,dz$

6) $\int\frac{x+3}{\sqrt{x^2+6x}}\,dx$

7) $\int\frac{x+2}{x^2}\,dx$

8) $\int\frac{e^x}{1+3e^x}\,dx$

9) $\int 3x^3\cdot e^{-2x^4}\,dx$

10) $\int 2^{-4x}\,dx$

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Sol. –

$$1)\ \int 3x^2\,dx = 3\int x^2 dx = 3\frac{x^3}{3} + c = x^3 + c$$

$$2)\ \left(x^{-2} + x\right)dx = \int x^{-2}\,dx + \int x\,dx = \frac{x^{-1}}{-1} + \frac{x^2}{2} + c = -\frac{1}{x} + \frac{x^2}{2} + c$$

$$3)\ \int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int 2x(x^2+1)^{1/2}dx = \frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2} + c = \frac{1}{3}\sqrt{(x^2+1)3} + c$$

$$4)\ \int\left(2t + t^{-1}\right)^2 dt = \int\left(4t^2 + 4 + t^{-2}\right)dt = 4\frac{t^3}{3} + 4t + \frac{t^{-1}}{-1} + c = \frac{4}{3}t^3 + 4t - \frac{1}{t} + c$$

$$5)\ \int\sqrt{(z^2 - z^{-2})^2 + 4}\,dz = \int\sqrt{z^4 - 2 + z^{-4} + 4}\,dz = \int\sqrt{z^4 + 2 + z^{-4}}\,dz$$

$$= \int\sqrt{(z^2 + z^{-2})^2}\,dz = \int(z^2 + z^{-2})\,dz = \frac{z^3}{3} + \frac{z^{-1}}{-1} + c = \frac{1}{3}z^3 - \frac{1}{z} + c$$

$$6)\ \int\frac{x+3}{\sqrt{x^2+6x}}\,dx = \frac{1}{2}\int(2x+6)\cdot(x^2+6x)^{-1/2}dx$$

$$= \frac{1}{2}\cdot\frac{(x^2+6x)^{1/2}}{1/2} + c = \sqrt{x^2+6x} + c$$

$$7)\ \int\frac{x+2}{x^2}\,dx = \int\left(\frac{x}{x^2} + \frac{2}{x^2}\right)dx = \int\left(x^{-1} + 2x^{-2}\right)dx = \ln x + \frac{2x^{-1}}{-1} + c = \ln x - \frac{2}{x} + c$$

$$8)\ \int\frac{e^x}{1+3e^x}\,dx = \frac{1}{3}\int 3e^x(1+3e^x)^{-1}dx = \frac{1}{3}\ln(1+3e^x) + c$$

$$9)\ \int 3x^3\cdot e^{-2x^4}\,dx = -\frac{3}{8}\int -8x^3\cdot e^{-2x^4}\,dx = -\frac{3}{8}\cdot e^{-2x^4} + c$$

$$10)\ \int 2^{-4x}dx = -\frac{1}{4}\int 2^{-4x}\cdot(-4dx) = -\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2} + c$$

Integrals of trigonometric functions:

The integration formulas for the trigonometric functions are:

6) $\int \sin u \cdot du = -\cos u + c$

7) $\int \cos u \cdot du = \sin u + c$

8) $\int \tan u \cdot du = -\ln|\cos u| + c$

9) $\int \cot u \cdot du = \ln|\sin u| + c$

10) $\int \sec u \cdot du = \ln|\sec u + \tan u| + c$

11) $\int \csc u \cdot du = -\ln|\csc u + \cot u| + c$

12) $\int \sec^2 u \cdot du = \tan u + c$

13) $\int \csc^2 u \cdot du = -\cot u + c$

14) $\int \sec u \cdot \tan u \cdot du = \sec u + c$

15) $\int \csc u \cdot \cot u \cdot du = -\csc u + c$

EX-2- Evaluate the following integrals:

1) $\int \cos(3\theta - 1)d\theta$

2) $\int x \cdot \sin(2x^2)\,dx$

3) $\int \cos^2(2y) \cdot \sin(2y)\,dy$

4) $\int \sec^3 x \cdot \tan x\ dx$

5) $\int \sqrt{2 + \sin 3t} \cdot \cos 3t\ dt$

6) $\int \frac{d\theta}{\cos^2 \theta}$

7) $\int (1 - \sin^2 3t) \cdot \cos 3t\ dt$

8) $\int \tan^3(5x) \cdot \sec^2(5x)\ dx$

9) $\int \sin^4 x \cdot \cos^3 x\ dx$

10) $\int \frac{\cot^2 \sqrt{x}}{\sqrt{x}} dx$

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Sol.-

1) $\frac{1}{3}\int 3\cos(3\theta-1)d\theta = \frac{1}{3}\sin(3\theta-1)+c$

2) $\frac{1}{4}\int 4x\cdot\sin(2x^2)dx = -\frac{1}{4}\cos(2x^2)+c$

3) $-\frac{1}{2}\int(\cos 2y)^2\cdot(-2\sin 2y\ dy) = -\frac{1}{2}\cdot\frac{(\cos 2y)^3}{3}+c = -\frac{1}{6}(\cos 2y)^3+c$

4) $\int \sec^2 x\cdot(\sec x\cdot\tan x\cdot dx) = \frac{\sec^3 x}{3}+c$

5) $\frac{1}{3}\int(2+\sin 3t)^{1/2}(3\cos 3t\ dt) = \frac{1}{3}\cdot\frac{(2+\sin 3t)^{3/2}}{3/2}+c = \frac{2}{9}\sqrt{(2+\sin 3t)^3}+c$

6) $\int\frac{d\theta}{\cos^2\theta} = \int\sec^2\theta\cdot d\theta = \tan\theta+c$

7) $\int(1-\sin^2 3t)\cdot\cos 3t\ dt = \frac{1}{3}\int 3\cos 3t\ dt - \frac{1}{3}\int(\sin 3t)^2\cdot 3\cos 3t\ dt$

$= \frac{1}{3}\sin 3t - \frac{1}{3}\cdot\frac{\sin^3 3t}{3}+c = \frac{1}{3}\cdot\sin 3t - \frac{1}{9}\sin^3 3t+c$

8) $\frac{1}{5}\int\tan^3 5x\cdot(5\sec^2 5x\ dx) = \frac{1}{5}\cdot\frac{\tan^4 5x}{4}+c = \frac{1}{20}\tan^4 5x+c$

9) $\int\sin^4 x\cdot\cos^3 x\ dx = \int\sin^4 x\cdot(1-\sin^2 x)\cdot\cos x\ dx$

$= \int\sin^4 x\cdot\cos x\ dx - \int\sin^6 x\cdot\cos x\ dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7}+c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\,dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\,dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x\,dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\,dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x\,dx$, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad (2)$$

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Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

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EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$\begin{aligned}
\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx &= \int (1 - \sin^2 x)^2 \, d(\sin x) && m = 0 \\
&= \int (1 - u^2)^2 \, du && u = \sin x \\
&= \int (1 - 2u^2 + u^4) \, du \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.
\end{aligned}$$

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EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x,$ $du = 2\cos 2x \, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

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Integrals of Powers of tan x and sec x

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx.\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

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EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\int \sec^3 x\, dx = \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx)$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos (m - n)x - \cos (m + n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin (m - n)x + \sin (m + n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos (m - n)x + \cos (m + n)x]. \tag{5}$$

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin (-2x) + \sin 8x] \, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

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Integrals of inverse trigonometric functions:

The integration formulas for the inverse trigonometric functions are:

$$16)\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \qquad ; \qquad \forall u^2 < a^2$$

$$17)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

$$18)\ \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \qquad ; \qquad \forall u^2 > a^2$$

EX-3 **Evaluate the following integrals:**

$1)\ \int \frac{x^2}{\sqrt{1-x^6}}\,dx$

$2)\ \int \frac{dx}{\sqrt{9-x^2}}$

$3)\ \int \frac{x}{1+x^4}\,dx$

$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$5)\ \int \frac{dx}{x\sqrt{4x^2-1}}$

$6)\ \int \frac{2dx}{\sqrt{x}(1+x)}$

$7)\ \int \frac{dx}{1+3x^2}$

$8)\ \int \frac{2\cos x}{1+\sin^2 x}\,dx$

$9)\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$10)\ \int \frac{\tan^{-1}x}{1+x^2}\,dx$

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Sol.-

$$1)\ \frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}\left(3x^2\,dx\right)=\frac{1}{3}\sin^{-1}x^3+c$$

$$2)\ \int \frac{dx}{\sqrt{9-x^2}}=\sin^{-1}\frac{x}{3}+c$$

$$3)\ \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx=\frac{1}{2}\tan^{-1}x^2+c$$

$$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx=\sin^{-1}(\tan x)+c$$

$$5)\ \int \frac{2\ dx}{2x\sqrt{(2x)^2-1}}=\sec^{-1}(2x)+c$$

$$6)\ \int \frac{2}{\sqrt{x}(1+x)}dx=4\int \frac{\frac{1}{2\sqrt{x}}\ dx}{1+(\sqrt{x})^2}=4\tan^{-1}\sqrt{x}+c$$

$$7)\ \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\ dx}{1+(\sqrt{3}x)^2}=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x)+c$$

$$8)\ 2\int \frac{\cos x\ dx}{1+(\sin x)^2}=2\tan^{-1}(\sin x)+c$$

$$9)\ \int e^{\sin^{-1}x}\cdot\frac{dx}{\sqrt{1-x^2}}=e^{\sin^{-1}x}+c$$

$$10)\ \int \tan^{-1}x\cdot\frac{dx}{1+x^2}=\frac{(\tan^{-1}x)^2}{2}+c$$

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Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

19) $\int \sinh u \cdot du = \cosh u + c$

20) $\int \cosh u \cdot du = \sinh u + c$

21) $\int \tanh u \cdot du = \ln(\cosh u) + c$

22) $\int \coth u \cdot du = \ln(\sinh u) + c$

23) $\int \sec h^2 u \cdot du = \tanh u + c$

24) $\int \csc h^2 u \cdot du = \coth u + c$

25) $\int \sec hu \cdot \tanh u \cdot du = -\sec hu + c$

26) $\int \csc hu \cdot \coth u \cdot du = -\csc hu + c$

EX-4 **– Evaluate the following integrals:**

1) $\int \frac{\cosh(\ln x)}{x} dx$

2) $\int \sinh(2x+1)\, dx$

3) $\int \frac{\sinh x}{\cosh^4 x} dx$

4) $\int x \cdot \cosh(3x^2)\, dx$

5) $\int \sinh^4 x \cdot \cosh x \; dx$

6) $\int \sec h^2(2x-3)\, dx$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

8) $\int \left(e^{ax} - e^{-ax}\right) dx$

9) $\int \frac{\sinh x}{1+\cosh x} dx$

10) $\int \operatorname{csch}^2 x \cdot \coth x\, dx$

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Sol.-

$$1)\ \int \cosh(\ln x)\cdot\left(\frac{dx}{x}\right)=\sinh(\ln x)+c$$

$$2)\ \frac{1}{2}\int \sinh(2x+1)\cdot(2\,dx)=\frac{1}{2}\cosh(2x+1)+c$$

$$3)\ \int \frac{1}{\cosh^3 x}\cdot\frac{\sinh x}{\cosh x}\ dx=\int \operatorname{sech}^3 x\cdot\tanh x\ dx$$

$$=-\int \operatorname{sech}^2 x\cdot(-\operatorname{sech} x\cdot\tanh x\ dx)=-\frac{\operatorname{sech}^3 x}{3}+c$$

$$4)\ \frac{1}{6}\int \cosh(3x^2)\cdot(6x\,dx)=\frac{1}{6}\sinh(3x^2)+c$$

$$5)\ \int \sinh^4 x\cdot(\cosh x\ dx)=\frac{\sinh^5 x}{5}+c$$

$$6)\ \frac{1}{2}\int \operatorname{sech}^2(2x-3)\cdot(2\ dx)=\frac{1}{2}\tanh(2x-3)+c$$

$$7)\ \int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx=\int \tanh x\ dx=\ln(\cosh x)+c$$

$$8)\ 2\int \frac{e^{ax}-e^{-ax}}{2}\,dx=\frac{2}{a}\int \sinh ax\ (a\,dx)=\frac{2}{a}\cosh ax+c$$

$$9)\ \int \frac{\sinh x\ dx}{1+\cosh x}=\ln(1+\cosh x)+c$$

$$10)\ -\int \operatorname{csch} x\cdot(-\operatorname{csch} x\cdot\coth x\ dx)=-\frac{\operatorname{csch}^2 x}{2}+c$$

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Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

27) $\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$

28) $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$

29) $\int \frac{du}{1-u^2} = \begin{Bmatrix} \tanh^{-1} u + c & if \ |u| < 1 \\ \coth^{-1} u + c & if \ |u| > 1 \end{Bmatrix} = \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + c$

30) $\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$

31) $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$

EX-4 – **Evaluate the following integrals:**

1) $\int \frac{dx}{\sqrt{1+4x^2}}$ 2) $\int \frac{dx}{\sqrt{4+x^2}}$ 3) $\int \frac{dx}{1-x^2}$

4) $\int \frac{dx}{x\sqrt{4+x^2}}$ 5) $\int \frac{\sec^2\theta \ d\theta}{\sqrt{\tan^2\theta - 1}}$ 6) $\int \tanh^{-1}(\ln\sqrt{x}) \cdot \frac{dx}{x(1-\ln^2\sqrt{x})}$

Sol.-

1) $\frac{1}{2}\int \frac{2 \ dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sinh^{-1} 2x + c$

2) $\int \frac{\frac{1}{2} \ dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1}\frac{x}{2} + c$

3) $\int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad if \quad |x| < 1$

$= \coth^{-1} x + c \quad if \quad |x| > 1$

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$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int \frac{{}^{1}\!/_{2}\ dx}{{}^{x}\!/_{2}\sqrt{1+\left({}^{x}\!/_{2}\right)^2}} = -\frac{1}{2}csc\,h^{-1}\left|{}^{x}\!/_{2}\right| + c$$

$$5)\ \int \frac{1}{\sqrt{tan^2\theta - 1}}\left(sec^2\theta\ \ d\theta\right) = cosh^{-1}(tan\,\theta) + c$$

$$6)\quad let\quad u = ln\sqrt{x} = \frac{1}{2}ln\,x \qquad du = \frac{1}{2x}dx$$

$$\int tanh^{-1}(ln\sqrt{x})\cdot\frac{dx}{x(1-ln^2\sqrt{x})} = \int tanh^{-1}u\cdot\frac{2\,du}{1-u^2}$$

$$= 2\frac{\left(tanh^{-1}u\right)^2}{2} + c = \left[tanh^{-1}(ln\sqrt{x})\right]^2 + c$$

Problems

Evaluate the following integrals:

$1)\ \int (x^2-1)\cdot(4-x^2)dx$ $\qquad$ $\left(ans.:\ \frac{5}{3}x^3 - \frac{1}{5}x^5 - 4x + c\right)$

$2)\ \int e^x\cdot sin\,e^x\ dx$ $\qquad$ $\left(ans.:\ -cose^x + c\right)$

$3)\ \int tan(3x+5)\,dx$ $\qquad$ $\left(ans.:\ -\frac{1}{3}ln\left|cos(3x+5)\right| + c\right)$

$4)\ \int \frac{cot(lnx)}{x}dx$ $\qquad$ $\left(ans.:\ ln\left|sin(lnx)\right| + c\right)$

$5)\ \int \frac{sinx + cosx}{cosx}dx$ $\qquad$ $\left(ans.:\ -ln\left|cosx\right| + x + c\right)$

$6)\ \int \frac{dx}{1+cosx}$ $\qquad$ $\left(ans.:\ -cotx + cscx + c\right)$

$7)\ \int cot(2x+1)\cdot csc^2(2x+1)\,dx$ $\qquad$ $\left(ans.:\ -\frac{1}{4}cot^2(2x+1) + c\right)$

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8) $\int \frac{dx}{\sqrt{1-9x^2}}$ (ans.: $\frac{1}{3}\sin^{-1}(3x)+c$)

9) $\int \frac{dx}{\sqrt{2-x^2}}$ (ans.: $\sin^{-1}\frac{x}{\sqrt{2}}+c$)

10) $\int e^{2x}\cdot \cosh e^{2x}\,dx$ (ans.: $\frac{1}{2}\sinh e^{2x}+c$)

11) $\int e^{\sin x}\cdot \cos x\,dx$ (ans.: $e^{\sin x}+c$)

12) $\int \frac{dx}{e^{3x}}$ (ans.: $-\frac{1}{3}e^{-3x}+c$)

13) $\int \frac{e^{\sqrt{x}}-1}{\sqrt{x}}dx$ (ans.: $2e^{\sqrt{x}}-2\sqrt{x}+c$)

14) $\int x\left(a+b\sqrt{3x}\right)dx$ where a,b constants (ans.: $\frac{1}{10}(5ax^2+4\sqrt{3}bx^{5/2})+c$)

15) $\int \frac{dx}{-1-x^2}$ (ans.: $-\tan^{-1}x+c$)

16) $\int \frac{\cos\theta\,d\theta}{1+\sin^2\theta}$ (ans.: $\tan^{-1}(\sin\theta)+c$)

17) $\int \frac{1}{x^2}\csc\frac{1}{x}\cot\frac{1}{x}\,dx$ (ans.: $\csc\frac{1}{x}+c$)

18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}dx$ (ans.: $\frac{3}{4}\sqrt[3]{(3x^2+2x+1)^2}+c$)

19) $\int \sin(\tan\theta)\cdot\sec^2\theta\,d\theta$ (ans.: $-\cos(\tan\theta)+c$)

20) $\int \sqrt{x^2-x^4}\,dx$ (ans.: $-\frac{1}{3}\sqrt{(1-x^2)^3}+c$)

21) $\int \frac{\sec^2 2x\;dx}{\sqrt{\tan 2x}}$ (ans.: $\sqrt{\tan 2x}+c$)

22) $\int (\sin\theta - \cos\theta)^2 \, d\theta$ (ans.: $\theta + \cos^2\theta + c$)

23) $\int \frac{y}{y^4+1} dy$ (ans.: $\frac{1}{2}\tan^{-1} y^2 + c$)

24) $\int \frac{dx}{\sqrt{x}(x+1)}$ (ans.: $2\tan^{-1}\sqrt{x} + c$)

25) $\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}} dt$ (ans.: $\frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}} + c$)

26) $\int \frac{dx}{x^{\frac{1}{3}}\sqrt{1+x^{\frac{4}{3}}}}$ (ans.: $\frac{5}{2}\sqrt{1+x^{\frac{4}{3}}} + c$)

27) $\int \frac{(\cos^{-1}4x)^2}{\sqrt{1-16x^2}} dx$ (ans.: $-\frac{1}{12}(\cos^{-1}4x)^3 + c$)

28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ (ans.: $\sec^{-1}(2x) + c$)

29) $\int \frac{dx}{(e^x+e^{-x})^2}$ (ans.: $\frac{1}{4}\tanh x + c$)

30) $\int 3^{\ln x^2} \frac{dx}{x}$ (ans.: $\frac{1}{2\ln 3} 3^{\ln x^2} + c$)

31) $\int \frac{\cot x \; dx}{\ln(\sin x)}$ (ans.: $\ln \ln(\sin x) + c$)

32) $\int \frac{(\ln x)^2}{x} dx$ (ans.: $\frac{1}{3}(\ln x)^3 + c$)

33) $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x} dx$ (ans.: $e^{\sec x} + c$)

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34) $\int \frac{dx}{x \cdot \ln x}$ (ans.: $\ln \ln x + c$)

35) $\int \frac{d\theta}{\cosh\theta + \sinh\theta}$ (ans.: $-e^{-\theta} + c$)

36) $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$ (ans.: $x - \frac{1}{5\ln 2} 2^{5x} + c$)

37) $\int \frac{e^{\tan^{-1} 2t}}{1 + 4t^2} dt$ (ans.: $\frac{1}{2} e^{\tan^{-1} 2t} + c$)

38) $\int \frac{\cot x}{\csc x} dx$ (ans.: $\sin x + c$)

39) $\int \sec^4 x \cdot \tan^3 x \; dx$ (ans.: $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + c$)

40) $\int \csc^4 3x \; dx$ (ans.: $-\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c$)

41) $\int \frac{\cos^3 t}{\sin^2 t} dt$ (ans.: $-\csc t - \sin t + c$)

42) $\int \frac{\sec^4 x}{\tan^4 x} dx$ (ans.: $-\frac{1}{3}\cot^3 x - \cot x + c$)

43) $\int \tan^2 4\theta \; d\theta$ (ans.: $\frac{1}{4}\tan 4\theta - \theta + c$)

44) $\int \frac{e^x}{1 + e^x} dx$ (ans.: $\ln(1 + e^x) + c$)

45) $\int \tan^3 2x \; dx$ (ans.: $\frac{1}{4}\tan^2 2x + \frac{1}{2}\ln|\cos 2x| + c$)

46) $\int \frac{\sec^2 x}{2 + \tan x} dx$ (ans.: $\ln(2 + \tan x) + c$)

47) $\int \sec^4 3x \; dx$ (ans.: $\frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x + c$)

48) $\int \frac{e^t}{1 + e^{2t}} dt$ (ans.: $\tan^{-1} e^t + c$)

49) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ (ans.: $2\sin\sqrt{x}+c$)

50) $\int \frac{dx}{\sin x\cdot\cos x}$ (ans.: $-\ln\left|\csc 2x+\cot 2x\right|+c$)

51) $\int \sqrt{1+\sin y}\;dy$ (ans.: $-2\sqrt{1-\sin y}+c$)

52) $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$ (ans.: $\ln(2+\tan^{-1}x)+c$)

53) $\int \sin^{-1}(\cosh x)\cdot\frac{\sinh x\;dx}{\sqrt{1-\cosh^2 x}}$ (ans.: $\frac{1}{2}\left(\sinh^{-1}(\cosh x)\right)^2+c$)

54) $\int \frac{\cos\theta\;d\theta}{1-\sin^2\theta}$ (ans.: $\ln\left|\sec\theta+\tan\theta\right|+c$)

55) $\int \frac{dx}{x\left(1+(\ln x)^2\right)}$ (ans.: $\tan^{-1}(\ln x)+c$)

56) $\int \left(e^{\frac{9}{4}x}-2e^{\frac{5}{4}x}+e^{\frac{x}{4}}\right)dx$ (ans.: $\frac{4}{9}e^{\frac{9}{4}x}-\frac{8}{5}e^{\frac{5}{4}x}+4e^{\frac{x}{4}}+c$)

57) $\int \frac{e^x\,dx}{e^{2x}+2e^x+1}$ (ans.: $-\frac{1}{e^x+1}+c$)

58) $\int e^x\cdot\sinh 2x\;dx$ (ans.: $\frac{1}{2}\left[\frac{1}{3}e^{3x}+e^{-x}\right]+c$)

59) $\int \frac{\sec^3 x+e^{\sin x}}{\sec x}\,dx$ (ans.: $\tan x+e^{\sin x}+c$)

60) $\int \frac{3^{x+2}}{2+9^{x+1}}\,dx$ (ans.: $\frac{3}{\sqrt{2}\ln 3}\tan^{-1}\frac{3^{x+1}}{\sqrt{2}}+c$)

61) $\int \frac{\cos x\;dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}$ (ans.: $2\sin^{-1}\sqrt{\sin x}+c$)

62) $\int \tan^5 x \; dx$ (ans.: $\frac{1}{4}\sec^4 x - \sec^2 x - \ln|\cos x| + c$)

63) $\int e^{\ln \sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}}$ (ans.: $\frac{1}{2}(\sin^{-1} x)^2 + c$)

64) $\int x \cdot e^{x^2-1} \; dx$ (ans.: $\frac{1}{2}e^{x^2-1} + c$)

65) $\int \cosh(\ln \cos x) \; dx$ (ans.: $\frac{1}{2}\left[\sin x + \ln|\sec x + \tan x|\right] + c$)

66) $\int \frac{\cos x}{\sin^2 x} dx$ (ans.: $-\csc x + c$)

67) $\int \cosh^{-1}(\sin x) \; \frac{\cos x \, dx}{\sqrt{\sin^2 x - 1}}$ (ans.: $\frac{1}{2}\left[\cosh^{-1}(\sin x)\right]^2 + c$)

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