Directional derivative.

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

هنا نجد تغير أو اشتقاق المتغيرات كلها في آن واحد .

$\vec{u} = (a, b)$ unit vactor

$D_{\vec{u}} f(x,y)$ directional derivative.

- The function of a single variable :-

$$g(z) = f(x_0 + az, y_0 + bz)$$

$$g'(z) = \lim_{h \to 0} \frac{g(z+h) - g(z)}{h}$$

at $z = 0$

$$\therefore g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

sub. in for $g(z)$:-

$$g'(0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

$$\therefore g'(0) = D_{\vec{u}} f(x_0, y_0)$$

$$\therefore D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$

$$D_{\vec{u}} f(x,y,z) = f_x(x,y,z)\, a + f_y(x,y,z)\, b + f_z(x,y,z)\, c$$

مشتقة بالنسبة لـ x

مشتقة بالنسبة لـ y

مشتقة بالنسبة لـ z

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$, or $\vec{u} = \dfrac{\vec{PQ}}{|PQ|}$

$PQ = (x_1 - x_0)i + (y_1 - y_0)j + (z_1 - z_0)K$

$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

EX 1 // Find Each directional derivatives :-

a/ $D_{\vec{u}} f(2,0)$ where $f(x,y) = xe^{xy} + y$, $\vec{u}$ is the unit vector of $\theta = \frac{2\pi}{3}$

$\vec{u} = (a, b)$

$\vec{u} = (\cos\theta, \sin\theta) = \left(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$

$D_{\vec{u}} f(x,y) = \left(-\frac{1}{2}\right)\underbrace{\left(e^{xy} - xye^{xy}\right)}_{x \text{ مشتقة}} + \left(\frac{\sqrt{3}}{2}\right)\underbrace{\left(x^2e^{xy} + 1\right)}_{y \text{ مشتقة}}$

$D_{\vec{u}} f(2,0) = \left(-\frac{1}{2}\right)\left(e^0 - 0e^0\right) + \left(\frac{\sqrt{3}}{2}\right)\left(2^2e^0 + 1\right)$

$D_{\vec{u}} f(2,0) = \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5)$

Gradient forormula

$$D_{\vec{u}} f(\vec{x}) = \nabla f \cdot \vec{u}$$

$\vec{X} = (x, y, z)$ or $\vec{X} = (x, y)$

$\nabla f = (f_x, f_y, f_z)$ or $\nabla f = (f_x, f_y)$

مشتقة x   مشتقة y   مشتقة z

or $\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Ex 1 //

a/ $D_{\vec{u}} f(\vec{x})$ for, $f(x, y) = x\cos(y)$ in direction of $\vec{v} = (2,1)$

$\nabla f (\cos y, -x\sin y)$

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$, $|\vec{v}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\therefore \vec{u} = \left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$

$D_{\vec{u}} f(\vec{x}) = (\cos y, -x\sin y) \cdot \left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$

dot product
عملية ضرب نقطي
بين متجهين
x مع x و y مع y

$D_{\vec{u}} f(\vec{x}) = \dfrac{2}{\sqrt{5}} \cdot \cos y + \left(-\dfrac{1}{\sqrt{5}} \cdot x\sin y\right)$

$= \dfrac{1}{\sqrt{5}} (2\cos y - x\sin y)$

b/ $D_{\vec{u}} f(\vec{x})$ for $f(x,y,z) = \sin(yz) + \ln(x^2)$ at $(1,1,\pi)$

$\vec{v} = (1,1,-1)$

$$\nabla f(x,y,z) = \left(\frac{2}{x}, z\cos(yz), y\cos yz\right)$$

$$\nabla f(1,1,\pi) = \left(\frac{2}{1}, \pi\cos(\pi), \cos(\pi)\right)$$

$$= (2, -\pi, -1)$$

$$|\vec{v}| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$$

$$\vec{u} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$$

$$D_{\vec{u}} f(1,1,\pi) = (2,-\pi,-1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$D_{\vec{u}} f(1,1,\pi) = \frac{2}{\sqrt{3}} - \frac{\pi}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}[2-\pi+1] = \frac{3-\pi}{\sqrt{3}}$$

c/ $f(x,y) = \cos\left(\frac{x}{y}\right)$, $\vec{v} = \langle 3,-4 \rangle$.

$$\nabla f = \left(-\frac{1}{y}\sin\frac{x}{y}, \frac{x}{y^2}\sin\frac{x}{y}\right)$$

$$|\vec{v}| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$$

$\therefore u = \frac{3}{5}, \frac{-4}{5}$

$$\therefore D_{\vec{u}} f = \left[-\frac{1}{y}\sin\frac{x}{y}, \frac{x}{y^2}\sin\frac{x}{y}\right] \cdot \left[\frac{3}{5}, -\frac{4}{5}\right]$$

$$D_{\vec{u}}f = -\frac{3}{5y}\sin\frac{x}{y} - \frac{4x}{5y^2}\sin\frac{x}{y}$$

$$= -\frac{1}{5}\left(\frac{3}{y} + \frac{4x}{y^2}\right)\sin\frac{x}{y}$$

d/ $f(x, y, z) = x^2y^3 - 4xz \quad , \quad \vec{v} = (-1, 2, 0)$

$$\nabla f = (2xy^3 - 4z, 3x^2y^2, -4x)$$

$$|\vec{v}| = \sqrt{(-1)^2 + 2^2 + 0} = \sqrt{5}$$

$$\vec{u} = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$D_{\vec{u}}f = (2xy^3 - 4z, 3x^2y^2, -4x)\cdot\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$D_{\vec{u}}f = \left(-\frac{(2xy^3 - 4z)}{\sqrt{5}} + \frac{6x^2y^2}{\sqrt{5}}\right)$$

$$D_{\vec{u}}f = \frac{1}{\sqrt{5}}\left(4z - 2xy^3 + 6x^2y^2\right)$$

Ex 2 //

$D_{\vec{u}}f(3, -1, 0)$ for $f(x, y, z) = 4x - y^2e^{3xz}$

$\vec{v} = (-1, 4, 2)$

H.W