Complex Numbers :- الاعداد المركبة

About 300 years ago, equations such as $x^2+1=0$ & $x^2+2x+4=0$ had no solution, where $x=\mp\sqrt{-1}$. Until the scientist Carl Friedrich Gauss (1777–1855) came with what called a complex number.

These new numbers were of the form $a+ib$, where a & b are real & i satisfies the equation $i^2=-1$ i.e $i=\sqrt{-1}$

It is important to understand that the plus sign in $a+ib$ does not denote addition: rather $a+ib$ is a single number, not the sum of a and ib.

∴ The typical standard Cartesian form of the complex number is

$$z=a+ib$$

a — real part number
b — imaginary part number.

$z=a+i0 \Rightarrow$ purly real complex no.
$z=0+ib \longrightarrow$ purly imaginary complex no.

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Definitions:- تعاريف

① If $z = a + ib$, then $\bar{z} = a - ib$ is complex conjugate of z

② If $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$, then

$z_1 \mp z_2 = (a_1 \mp a_2) + i(b_1 \mp b_2)$

③ If $z_3 = a_3 + ib_3$ then;

* $z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$ — تبادلي (commutative)
* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$,
* $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ — ترابطي (associative)
* $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ — توزيعي (distributive)

④ $z_1 * z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

⑤ $z + (0 + i0) = z$; $(1 + i0) z = z$

i.e there are zero & unit complex numbers.

$0 = 0 + i0$

$1 = 1 + i0$

⑥ $z_1 = z_2$ if & only if $a_1 = a_2$ & $b_1 = b_2$

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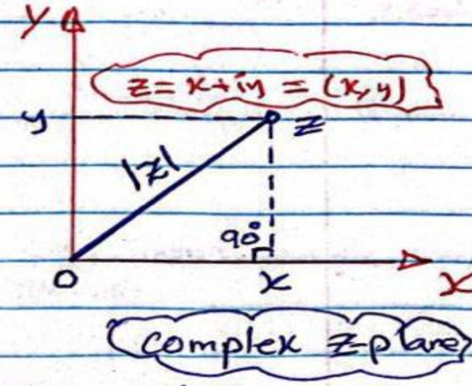
Complex plane : المستوي المركب

We can represent $z = a + ib$ as a point in a Cartesian a, b plane. And since Cartesian axes are more usually denoted by x and y rather than by a and b, let us write

$$z = x + iy = (x, y)$$

and represent z as a point in a so called "complex z plane"



* The x-axis is called the "real axis"
* The y-axis is called the "imaginary axis"
* From the definition ②, the addition of the complex number satisfies the parallelogram law for the addition of vectors, so that the it is often convenient to think of complex numbers as vectors.
* The distance from the origin to the point z (i.e., the "length of the z vector") is called the modulus of z, and denoted as $|z|$

$$|z| = \sqrt{x^2 + y^2} = \text{Modulus of } z$$

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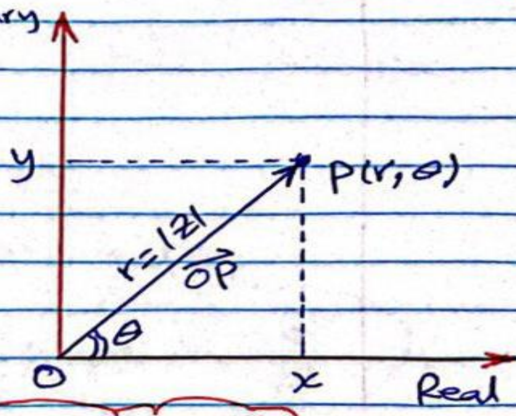
Polar Form of Complex number :- الشكل القطبي للأعداد المركبة

The vector $\vec{op}$ from the origin to $p(r,\theta)$ which is called polar form of complex number, where r & θ are called "Polar coordinates"



$$z = x + iy = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$; θ = argument angle with x-axis

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ ← "Euler's Formula"

Ex1) Proof that $e^{i\theta} = \cos\theta + i\sin\theta$?

Solution

By using a Taylor Series of $e^{i\theta}$ yields

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ Ans

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Ex) ① proof that: $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Solution

$e^{i\theta} = \cos\theta + i\sin\theta$ — ① ← Euler formula

$e^{-i\theta} = \cos\theta - i\sin\theta$ — ②

By adding eqs ① & ②, yields;

$e^{i\theta} + e^{-i\theta} = 2\cos\theta \longrightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

② proof that: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Solution

By subtracting eqs ② from ①; yields,

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Longrightarrow \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

More Definitions:-

If $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and,

$z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$ then,

⑦ $k \cdot z_1 = k(x_1 + iy_1) = kx_1 + i\,ky_1 = kr_1 e^{i\theta_1}$

⑧ $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = r_1r_2 e^{i(\theta_1+\theta_2)}$

⑨ $z_1 \cdot \bar{z}_1 = |z_1|^2$ & $z_2 \cdot \bar{z}_2 = |z_2|^2$

(10) $$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

(11) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \quad ; \quad r_2 \neq \text{zero}$$

(12) $$\bar{\bar{z}}_1 = z_1 \quad \& \quad \bar{\bar{z}}_2 = z_2$$

Ex) Evaluate $\frac{2+i}{3-4i}$

Solution

$$\frac{2+i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{(6-4)+(8+3)i}{9+16} = \frac{2}{25} + \frac{11}{25} i$$

Ex) let $z_1 = 2+i3$ & $z_2 = 4+i$

Find; ① $z_1 + z_2$

② $z_1 - z_2$

③ $z_1 \cdot z_2$

Solution

① $z_1 + z_2 = (2+i3) + (4+i) = (2+4) + (3+1)i = \boxed{6+i4}$

② $z_1 - z_2 = (2+i3) - (4+i) = (2-4) + (3-1)i = \boxed{-2+i2}$

③ $z_1 \cdot z_2 = (2+i3) \cdot (4+i) = 2\times4 + i2 + 3\times4i + 3\times1\times i\times i$

$= 8 + i2 + i12 + 3i^2 = 8 - 3 + i14$

$= \boxed{5+i14}$; $i^2 = -1$

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Ex) put the Complex number $(1-i\sqrt{3})$ in the Polar Form?

Solution

$$Z = 1 - i\sqrt{3} = x + iy = |Z|\, e^{i\theta} = r\, e^{i\theta}$$

$$|Z| = r = \sqrt{x^2+y^2} = \sqrt{1^2+(-\sqrt{3})^2} = \sqrt{4} = \boxed{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{1} = \tan^{-1}(-\sqrt{3}) = \boxed{-60^\circ}$$

$$= \boxed{300^\circ} = \boxed{\frac{5\pi}{3}}$$

$$\therefore Z = r\, e^{i\theta} = r(\cos\theta + i\sin\theta)$$

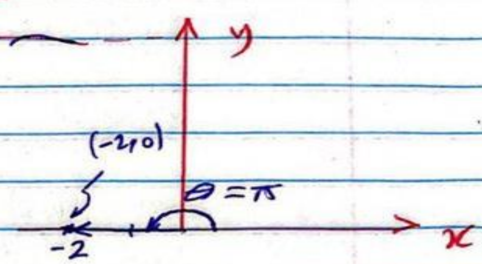
$$= \boxed{2\, e^{i\frac{5\pi}{3}} = 2(\cos 300 + i\sin 300)}$$ Ans

Note:

$$\ln r e^{i\theta} = \ln r + i\theta$$

Ex) Find $\ln(-2)$?

Solution



$$\ln -2 = \ln(-2 + i0)$$

$$r = \sqrt{(-2)^2 + (0)^2} = \sqrt{4} = \boxed{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{-2} = \tan^{-1}0 = \boxed{0}$$

Here we need to correct the angle by add π

$$\therefore \boxed{\theta = \pi}$$

$$\therefore \boxed{\ln(-2) = \ln 2 + i\pi}$$ Ans

Ex) Find x, y if $(3+i4)^2 - 2(x - iy) = x + iy$

Solution

$$9 + i \times 2 \times 3 \times 4 + (i4)^2 - 2x + i2y = x + iy$$

$$9 + i24 + \cancel{i^2}^{-1} 16 - 2x + i2y = x + iy$$

$$9 + i24 - 16 - 2x + i2y = x + iy$$

$$-7 - 2x + i(24 + 2y) = x + iy$$

* From Def # 6

$\therefore -7 - 2x = x \longrightarrow -7 = 3x \longrightarrow \boxed{x = -7/3}$ Ans

& $24 + 2y = y \longrightarrow \boxed{y = -24}$ Ans

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De Moivre's Theorem :- نظرية ديموفر

If n is any integer (positive, negative, or zero), and z is a complex number in a polar form; then,

$$z^n = (r e^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

This equation is well Known as "De Moviers formula"

Note

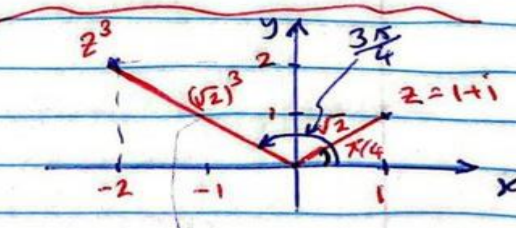
In complex plane θ is not uniquely determined at any given point z, So, if we put $\theta = \theta_0 + 2\pi k$ into the power polar form gives

$$z^n = r^n e^{in(\theta_0 + 2\pi k)} = r^n e^{in\theta_0} e^{i2\pi kn}$$

Then, $e^{i2\pi kn} = \cos(2\pi kn) + i \sin(2\pi kn) = 1 + i0$
$= 1$ for all integer K.

So, $z^n = r^n e^{in\theta_0}$ and $\boxed{\theta_0 = \theta}$

Ex) compute $(1+i)^3$?



Solution

$\boxed{n=3}$

$z = 1+i = a+ib$

$\left.\begin{matrix} a=1 \\ b=1 \end{matrix}\right\} \Longrightarrow |z| = r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \boxed{\sqrt{2}}$

$\theta = \tan^{-1}\frac{b}{a} \Longrightarrow \theta = \tan^{-1}\frac{1}{1} = \boxed{45^\circ} = \boxed{\frac{\pi}{4}}$

$z^n = r^n(\cos n\theta + i \sin n\theta) = (\sqrt{2})^3(\cos\frac{3}{4}\pi + i \sin\frac{3\pi}{4})$
$= 2^{3/2}(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \boxed{-2+2i}$

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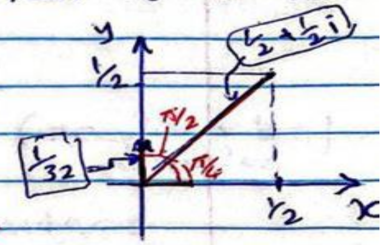
We can also solve the previous example by:-

$(1+i)^3 = (1+i)(1+i)(1+i) = (1+i)(2i) = \boxed{2i-2}$, but it is gonna be quite exhausted if n is large.



Ex) Compute $\left(\frac{1}{2} + i\frac{1}{2}\right)^{10}$?

(Solution)

$\boxed{n=10}$; $a = \frac{1}{2}$, $b = \frac{1}{2}$

$$r = \sqrt{a^2+b^2} = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} = \boxed{\frac{1}{\sqrt{2}}}$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{1/2}{1/2} = \boxed{45^\circ} = \boxed{\frac{\pi}{4}}$$

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{10}\left(\cos \frac{\cancel{10}^{5}}{\cancel{4}_{2}}\pi + i \sin \frac{\cancel{10}^{5}}{\cancel{4}_{2}}\pi\right)$$

$$= \frac{1}{32}(0 + i) = \boxed{\frac{1}{32} i} \leftarrow \boxed{\theta = \frac{\pi}{2}}$$

Fractional Powers :-

If we consider the function $z^{1/n}$, which called the n^{th} root of z, we write;

$$z^{1/n} = \left(r\, e^{i(\theta_0 + 2\pi k)}\right)^{1/n} = r^{1/n} e^{i\left(\frac{\theta_0 + 2\pi k}{n}\right)}$$

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta_0 + 2\pi k}{n}\right) + i \sin\left(\frac{\theta_0 + 2\pi k}{n}\right)\right] = F_k$$

$k = 0, 1, 2, 3, \ldots, n-1$

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Ex) Find the value of $(-1+i)^{1/3}$?

Solution

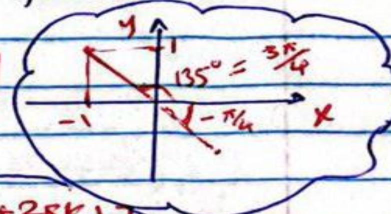
$n = 3 \quad , \quad K = 0, 1, 2$

$r = \sqrt{a^2+b^2} = \sqrt{(-1)^2+(1)^2} = \boxed{\sqrt{2}}$

$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{1}{-1} = \boxed{-45^\circ}$ {need to be corrected}

the complex no. is in 2nd quarter:

$\therefore \theta = 180 - 45 = \boxed{135^\circ} = \boxed{\frac{3\pi}{4}}$



$$Z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2\pi K}{n}\right) + i\sin\left(\frac{\theta+2\pi K}{n}\right)\right]$$

$$(-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{\frac{3\pi}{4}+2\pi K}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4}+2\pi K}{3}\right)\right]$$

① At $K=0 \longrightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$$= \boxed{(\sqrt{2})^{1/3}\left[\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right]}$$

② At $K=1 \longrightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{\frac{3\pi}{4}+2\pi}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4}+2\pi}{3}\right)\right]$

$$= (\sqrt{2})^{1/3}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]$$

$$= \boxed{(\sqrt{2})^{1/3}\left[-0.966 + i\,0.2588\right]}$$

③ At $K=2 \longrightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{\frac{3\pi}{4}+4\pi}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4}+4\pi}{3}\right)\right]$

$$= (\sqrt{2})^{1/3}\left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right]$$

$$= \boxed{(\sqrt{2})^{1/3}\left[0.2588 - 0.966\,i\right]}$$

Ex/ find the sixth roots of $z = -8$?

Solution

$z = -8 + 0i = a + ib \quad ; \quad a = -8, \; b = 0$

$r = \sqrt{a^2 + b^2} = \sqrt{(-8)^2 + 0^2} = \boxed{8}$

$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{-8} = \tan^{-1} 0 = 0$ (need correction)

$\theta = \pi + \text{zero} = \boxed{\pi}$

$n = 6$

$K = 0, 1, 2, 3, 4, 5$

$$z^{1/6} = r^{1/6}\left[\cos\left(\frac{\theta + 2\pi K}{6}\right) + i \sin\left(\frac{\theta + 2\pi K}{6}\right)\right]$$

① at $K = 0 \longrightarrow F_0 = 8^{1/6}\left[\cos\left(\frac{\pi + 2\pi(0)}{6}\right) + i \sin\left(\frac{\pi + 2\pi(0)}{6}\right)\right]$

$= 8^{1/6}\left[\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right]$

$= \boxed{\sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)}$

② at $K = 1 \longrightarrow F_1 = 8^{1/6}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \boxed{\sqrt{2}\, i}$

③ at $K = 2 \longrightarrow F_2 = 8^{1/6}\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right) = \boxed{\sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{1}{2} i\right)}$

④ at $K = 3 \longrightarrow F_3 = 8^{1/6}\left(\cos\frac{7\pi}{6} + i \sin\frac{7\pi}{6}\right) = \boxed{\sqrt{2}\left(\frac{-\sqrt{3}}{2} - \frac{1}{2} i\right)}$

⑤ at $K = 4 \longrightarrow F_4 = 8^{1/6}\left(\cos\frac{9\pi}{6} + i \sin\frac{9\pi}{6}\right) = \boxed{-\sqrt{2}\, i}$

⑥ at $K = 5 \longrightarrow F_5 = 8^{1/6}\left(\cos\frac{11\pi}{6} + i \sin\frac{11\pi}{6}\right) = \boxed{\sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)}$

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Ex// Find the roots of the equation $z^3 = -1 + i$?

Solution

$z^3 = -1 + i \longrightarrow z = (-1+i)^{1/3}$; $a = -1$, $b = 1$

$\therefore n = 3$, $k = 0, 1, 2$

$r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + 1^2} = \boxed{\sqrt{2}}$

$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{1}{-1} = -\frac{\pi}{4}$ (need correction)

$\therefore \theta = \boxed{\frac{3\pi}{4}} = \boxed{135°}$

$(-1+i)^{1/3} = r^{1/3} \left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i \sin\left(\frac{\theta + 2\pi k}{3}\right) \right]$

① at $k = 0 \longrightarrow r_0 = (\sqrt{2})^{1/3} \left[\cos\left(\frac{\frac{3\pi}{4} + 2\pi(0)}{3}\right) + i \sin\left(\frac{\frac{3\pi}{4} + 2\pi(0)}{3}\right) \right]$

$= (\sqrt{2})^{1/3} \left[\cos \frac{3\pi}{12} + i \sin \frac{3\pi}{12} \right] = \boxed{(\sqrt{2})^{1/3} (1 + i)}$

② at $k = 1 \longrightarrow r_1 = (\sqrt{2})^{1/3} \left[\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right]$

③ at $k = 2 \longrightarrow r_2 = (\sqrt{2})^{1/3} \left[\cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12} \right]$

Cauchy-Riemann Equations: معادلات كوشي ريمن

Theorem: نظرية

If $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at a point $z = x+iy$, then the 1st order partial derivatives of u & v are exist & satisfy the following at that point:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \&$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\longleftarrow$ Cauchy-Riemann Eqs.

To reproduce this theorem, we do the follows;

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ; \quad z = x+iy, \quad \Delta z = \Delta x + i\,\Delta y$$

* let us 1st set $\Delta y = 0$ in $\Delta z = \Delta x + i\Delta y$, so that $\Delta z = \Delta x$, i.e. $(z+\Delta z)$ approaches z parallel to the x-axis.

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + i\,v(x+\Delta x, y)] - [u(x,y) + i\,v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad (1)$$

* Alternatively, let us set $\Delta x = 0$ in $\Delta z = \Delta x + i\Delta y$, so that $\Delta z = i\Delta y$, i.e. $(z+\Delta z)$ approaches z parallel to the y-axis.

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$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + i v(x, y+\Delta y)] - [u(x,y) + i v(x,y)]}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)]}{i \Delta y} + \cancel{i} \lim_{\Delta y \to 0} \frac{[v(x, y+\Delta y) + v(x,y)]}{\cancel{i} \Delta y}$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad (2)$$

thus, the eq=s (1) & (2) must be equal to have $f'(z)$ exist at z, so,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Ex) Examine weither or not the eq= $f(z) = |z|^2$ differentiable ??

Solution

$f(z) = |z|^2 \quad ; \quad z = x + iy \longrightarrow |z| = \sqrt{x^2+y^2}$

$\therefore f(z) = (x^2+y^2) + i0$

$\therefore u = x^2 + y^2 \longrightarrow \frac{\partial u}{\partial x} = 2x \ , \ \frac{\partial u}{\partial y} = 2y$

$v = 0 \longrightarrow \frac{\partial v}{\partial x} = 0 \ , \ \frac{\partial v}{\partial y} = 0$

$\therefore$ just at the origin $(0,0)$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ &

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

so, this eq= is differentiable only at $z=0$

Ex) Show that $f(z) = z^2$ satisfies Cauchy–Riemann equations, & find the derivative in terms of the partial derivative?

Solution)

$f(z) = z^2 \quad ; \quad z = x + iy$

$\therefore f(z) = (x+iy)^2 = (x^2 - y^2) + i\,2xy$

so,

$u(x,y) = x^2 - y^2 \longrightarrow \frac{\partial u}{\partial x} = 2x \;;\; \frac{\partial u}{\partial y} = -2y$

$v(x,y) = 2xy \longrightarrow \frac{\partial v}{\partial x} = 2y \;;\; \frac{\partial v}{\partial y} = 2x$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \underline{\underline{2x}} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \underline{-2y}$

$\therefore$ The Cauchy–Riemann eq=s are satisfied & the Fun is differentiable.

To find the derivative then;

$$\boxed{f'(z) = u_x + i v_x}$$

$$f'(z) = 2x + i2y = 2\underbrace{(x+iy)}_{z} = \boxed{2z}$$

Cauch Riemann Eq=s in Polar Form:

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$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

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Ex) Show that $f(z)=\frac{1}{z}$ differentiable using Cauchy-Riemann eqs in polar form & find its derivative?

(Solution)

$f(z)=\frac{1}{z}$; $z=re^{i\theta}$

$f(z)=\frac{1}{re^{i\theta}}=\frac{1}{r}e^{-i\theta}=\frac{1}{r}(\cos\theta-i\sin\theta)$

$\therefore u(r,\theta)=\frac{1}{r}\cos\theta \rightarrow \frac{\partial u}{\partial r}=-\frac{1}{r^2}\cos\theta$; $\frac{\partial u}{\partial\theta}=\frac{-1}{r}\sin\theta$

$v(r,\theta)=-\frac{1}{r}\sin\theta \rightarrow \frac{\partial v}{\partial r}=\frac{1}{r^2}\sin\theta$; $\frac{\partial v}{\partial\theta}=\frac{-1}{r}\cos\theta$

$\therefore r\frac{\partial u}{\partial r}=-\frac{1}{r}\cos\theta=\frac{\partial v}{\partial\theta}$

& $\frac{\partial u}{\partial\theta}=-\frac{1}{r}\sin\theta=-r\frac{\partial v}{\partial r}$

} Cauchy-Riemann eqs are satisfied

$\therefore$ The fun is differentiable

$\therefore \boxed{f'(z)=e^{-i\theta}(u_r+iv_r)}$

$\therefore f'(z)=e^{-i\theta}\left(-\frac{1}{r^2}\cos\theta+i\frac{1}{r^2}\sin\theta\right)$

$=-\frac{1}{r^2}e^{-i\theta}\underbrace{(\cos\theta-i\sin\theta)}_{e^{-i\theta}}=\frac{-1}{r^2}e^{-i\theta}\cdot e^{-i\theta}$

$=\frac{-1}{r^2}e^{-i2\theta}=\frac{-1}{r^2e^{i2\theta}}=\frac{-1}{(\underbrace{re^{i\theta}}_{z})^2}=\boxed{\frac{-1}{z^2}}$

Example for discussions:
check the differentiability by Cauchy-Riemann eqs & drive it; $f(z)=e^z$

Complex numbers, Plar Formulam Euler equ, Power and Roots, Complex Func, Cautchy-Reiman Eqs -- نهاية محاضرة " الاعداد المركبة، الشكل القطبي، معادلو أويلر، قوى وجذور الاعداد المركبة، دالة العدد المركب، معادلات كوشي ريمن "--

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