Heat conduction through a plane wall

Case I : Uniform thermal conductivity

Refer to Fig. 2.4 (*a*) Consider a plane wall of homogeneous material through which heat is flowing *only in x-direction.*

Let, L = Thickness of the plane wall,

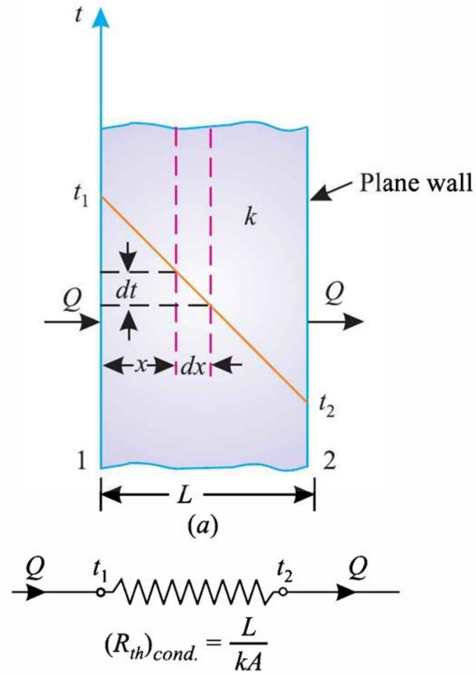
A = Cross-sectional area of the wall,

k = Thermal conductivity of the wall material, and

t_1, t_2 = Temperatures maintained at the two faces 1 and 2 of the wall, respectively.

The general heat conduction equation in cartesian coordinates is given by

$$\frac{\partial^2 t}{dx^2} + \frac{\partial^2 t}{dy^2} + \frac{\partial^2 t}{dz^2} + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \text{...[Eqn. 2.8]}$$



(*b*)

Fig. 2.4. Heat conduction through a plane wall.

If the heat conduction takes place under the conditions, steady state $\left(\frac{\partial t}{\partial \tau} = 0\right)$, one-dimensional $\left[\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0\right]$ and with no internal heat generation $\left(\frac{q_g}{k} = 0\right)$ then the above equation is reduced to

$$\frac{\partial^2 t}{dx^2} = 0, \quad \text{or} \quad \frac{d^2 t}{dx^2} = 0 \quad \text{...(2.33)}$$

By integrating the above differential twice, we have

$$\frac{dt}{dx} = C_1 \quad \text{and} \quad t = C_1 x + C_2 \quad \text{...(2.34)}$$

where C_1 and C_2 are the arbitrary constants. The values of these constants may be calculated from the known boundary conditions as follows :

At $x = 0$ $\quad t = t_1$

At $x = L$ $\quad t = t_2$

Substituting the values in the eqn. (2.34), we get

$$t_1 = O + C_2 \quad \text{and} \quad t_2 = C_1 L + C_2$$

After simplification, we have, $C_2 = t_1$ and $C_1 = \frac{t_2 - t_1}{L}$

Thus, the eqn. (2.34) reduces to :

$$t = \left(\frac{t_2 - t_1}{L}\right) x + t_1 \quad \text{...(2.35)}$$

The eqn. (2.35) indicates that *temperature distribution across a wall is linear* and is *independent of thermal conductivity*. Now heat through the plane wall can be found by using Fourier's equation as follows :

$$Q = -kA\frac{dt}{dx} \quad \left(\text{where, } \frac{dt}{dx} = \text{Temperature gradient}\right) \quad \ldots \text{[Eqn.(1.1)]}$$

But, $$\frac{dt}{dx} = \frac{d}{dx}\left[\left(\frac{t_2 - t_1}{L}\right)x + t_1\right] = \frac{t_2 - t_1}{L}$$

$$\therefore \quad Q = -kA\frac{(t_2 - t_1)}{L} = \frac{kA(t_1 - t_2)}{L} \quad \ldots(2.36)$$

Eqn (2.36) can be written as :

$$Q = \frac{(t_1 - t_2)}{(L/kA)} = \frac{(t_1 - t_2)}{(R_{th})_{cond.}} \quad \ldots(2.37)$$

where, $(R_{th})_{cond.}$ = Thermal resistance to heat conduction. Fig. 2.4 (*b*) shows the *equivalent thermal circuit* for heat flow through the plane wall.

Heat conduction through a composite wall

Refer to Fig. 2.6 (*a*). Consider the transmission of heat through a composite wall consisting of a number of slabs.

Let, L_A, L_B, L_C = Thicknesses of slabs *A*, *B* and *C* respectively (also called path lengths),

k_A, k_B, k_C = Thermal conductivities of the slabs *A*, *B*, and *C* respectively,

$t_1, t_4\ (t_1 > t_4)$ = Temperatures at the wall surfaces 1 and 4 respectively, and

t_2, t_3 = Temperatures at the interfaces 2 and 3 respectively.

Since the quantity of heat transmitted per unit time through each slab/layer is same, we have,

$$Q = \frac{k_A . A\,(t_1 - t_2)}{L_A} = \frac{k_B\,.\,A\,(t_2 - t_3)}{L_B} = \frac{k_C\,.\,A\,(t_3 - t_4)}{L_C}$$

(Assuming that there is a perfect contact between the layers and no temperature drop occurs across the interface between the materials).

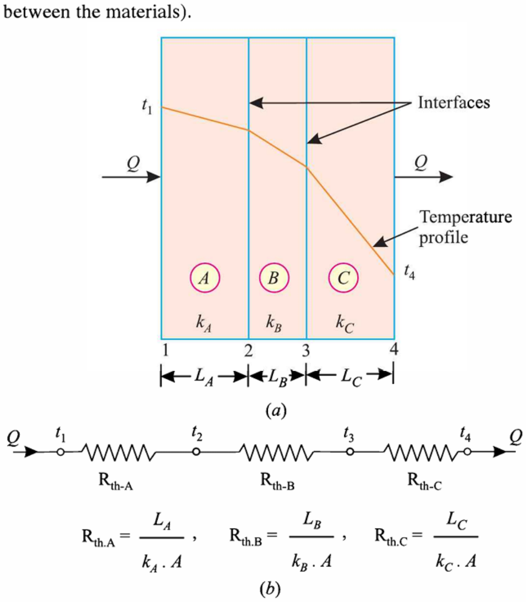


Fig. 2.6. Steady state conduction through a composite wall.

Rearranging the above expression, we get

$$t_1 - t_2 = \frac{Q \cdot L_A}{k_A \cdot A} \quad ...(i)$$

$$t_2 - t_3 = \frac{Q \cdot L_B}{k_B \cdot A} \quad ...(ii)$$

$$t_3 - t_4 = \frac{Q \cdot L_c}{k_C \cdot A} \quad ...(iii)$$

Adding (*i*), (*ii*) and (*iii*), we have

$$(t_1 - t_4) = Q\left[\frac{L_A}{k_A \cdot A} + \frac{L_B}{k_B \cdot A} + \frac{L_C}{k_C \cdot A}\right]$$

or,

$$Q = \frac{A\,(t_1 - t_4)}{\left[\dfrac{L_A}{k_A} + \dfrac{L_B}{k_B} + \dfrac{L_C}{k_C}\right]} \quad ...(2.49)$$

or,

$$Q = \frac{(t_1 - t_4)}{\left[\dfrac{L_A}{k_A \cdot A} + \dfrac{L_B}{k_B \cdot A} + \dfrac{L_C}{k_C \cdot A}\right]} = \frac{(t_1 - t_4)}{[R_{th-A} + R_{th-B} + R_{th-C}]} \quad ...[2.49\,(a)]$$

If the composite wall consists of *n* slabs/layers, then

$$Q = \frac{[t_1 - t_{(n+1)}]}{\sum_{1}^{n} \dfrac{L}{kA}} \quad ...(2.50)$$

In order to solve more complex problems involving both series and parallel thermal resistances, the electrical analogy may be used. A typical problem and its analogous electric circuit are shown in Fig. 2.7.

$$Q = \frac{\Delta t_{\text{overall}}}{\sum R_{th}} \quad ...(2.51)$$

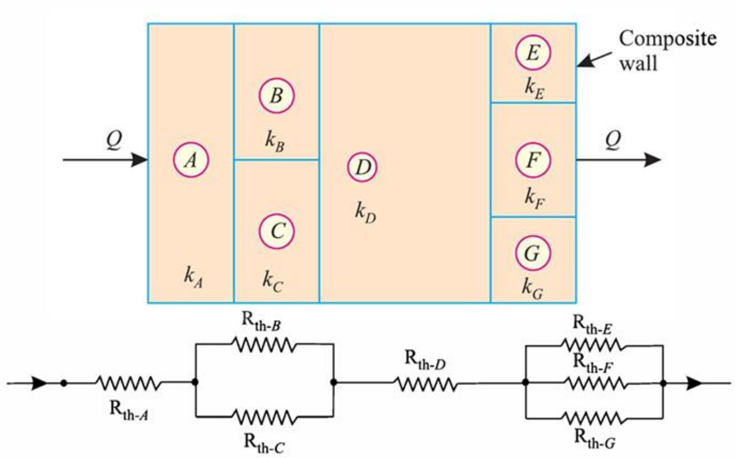


Fig. 2.7. Series and parallel one-dimensional heat transfer through a composite wall and electrical analog.

Thermal contact resistance

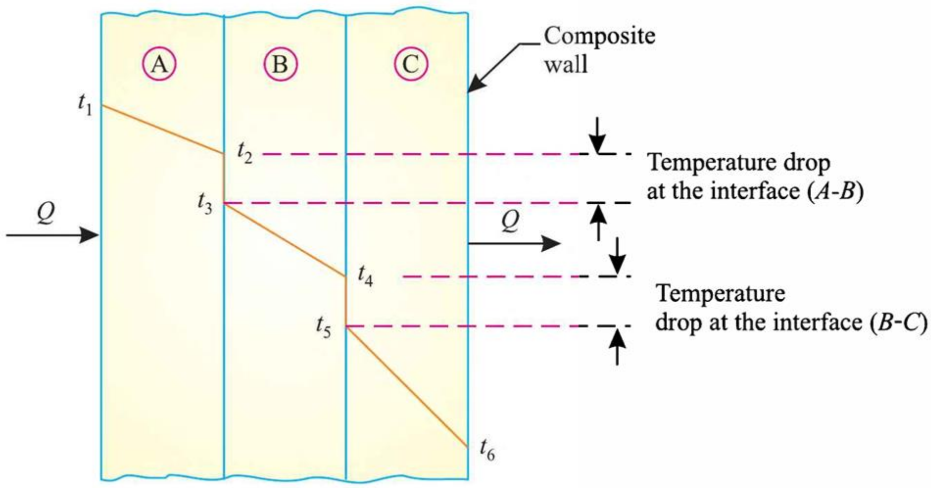


Fig. 2.8. Temperature drops at the interfaces.

Refer to Fig. 2.8. The contact resistances are given by

$$(R_{th-AB})_{cont.} = \frac{(t_2 - t_3)}{Q/A} \quad \text{and} \quad (R_{th-BC})_{cont.} = \frac{(t_4 - t_5)}{Q/A}$$

The overall heat transfer coefficient

While dealing with the problems of fluid to fluid heat transfer across a metal boundary, it is usual to adopt an overall heat transfer coefficient U which *gives the heat transmitted per unit area per unit time per degree temperature difference between the bulk fluids on each side of the metal.*

Refer to Fig. 2.9

Let,
L = Thickness of the metal wall,
k = Thermal conductivity of the wall material,
t_1 = Temperature of the surface–1,
t_2 = Temperature of the surface–2,
t_{hf} = Temperature of the hot fluid,
t_{cf} = Temperature of the cold fluid,
h_{hf} = Heat transfer coefficient from hot fluid to metal surface, and
h_{cf} = Heat transfer coefficient from metal surface to cold fluid.

(The suffices *hf* and *cf* stand for hot fluid and cold fluid respectively.)

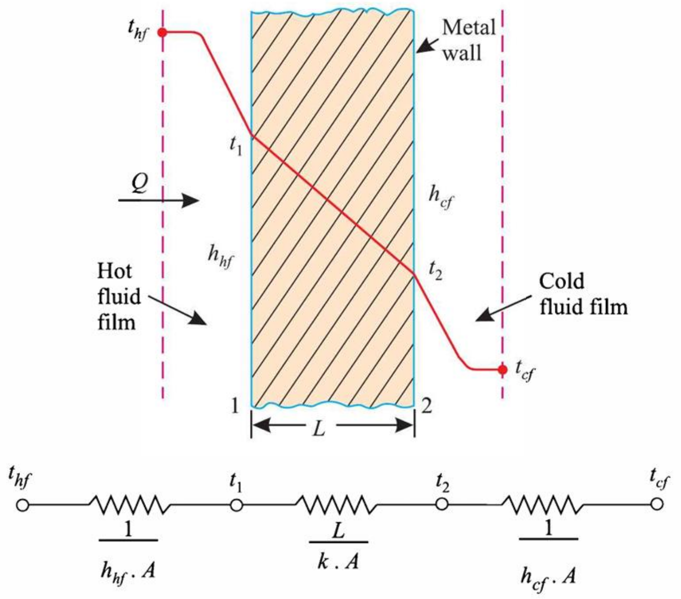


Fig. 2.9. The overall heat transfer through a plane wall.

The equations of heat flow through the fluid and the metal surface are given by

$$Q = h_{hf} A (t_{hf} - t_1) \qquad ...(i)$$

$$Q = \frac{k.A\,(t_1 - t_2)}{L} \qquad ...(ii)$$

$$Q = h_{cf} A (t_2 - t_{cf}) \qquad ...(iii)$$

By rearranging (*i*), (*ii*) and (*iii*), we get

$$t_{hf} - t_1 = \frac{Q}{h_{hf}\ .A} \qquad ...(iv)$$

$$t_1 - t_2 = \frac{QL}{k\ .A} \qquad ...(v)$$

$$t_2 - t_{cf} = \frac{Q}{k_{cf}\ .A} \qquad ...(vi)$$

Adding (*iv*), (*v*) and (*vi*) we get

$$t_{hf} - t_{cf} = Q\left[\frac{1}{h_{hf}\ .\ A} + \frac{L}{k\ .\ A} + \frac{1}{h_{cf}\ .\ A}\right]$$

or,

$$Q = \frac{A\,(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}} \qquad ...(2.52)$$

If U is the overall coefficient of heat transfer, then

$$Q = U.A\,(t_{hf} - t_{cf}) = \frac{A\,(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}}$$

or,

$$U = \frac{1}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}} \qquad ...(2.53)$$

Tutorial on heat conduction equation

Example 2.2. *The inner surface of a plane brick wall is at 60°C and the outer surface is at 35°C. Calculate the rate of heat transfer per m² of surface area of the wall, which is 220 mm thick. The thermal conductivity of the brick is 0.51 W/m°C.*

(AMIE Winter, 2000)

Solution. Temperature of the inner surface of the wall, $t_1 = 60°C$

Temperature of the outer surface of the wall, $t_2 = 35°C$

The thickness of the wall, $L = 220$ mm $= 0.22$ m

Thermal conductivity of the brick,

$k = 0.51$ W/m°C

Rate of heat transfer per m², q :

Rate of heat transfer per unit area,

$$q = \frac{Q}{A} = \frac{k\,(t_1 - t_2)}{L}$$

or

$$q = \frac{0.51 \times (60 - 35)}{0.22} = \mathbf{57.95\ W/m^2\ (Ans.)}$$

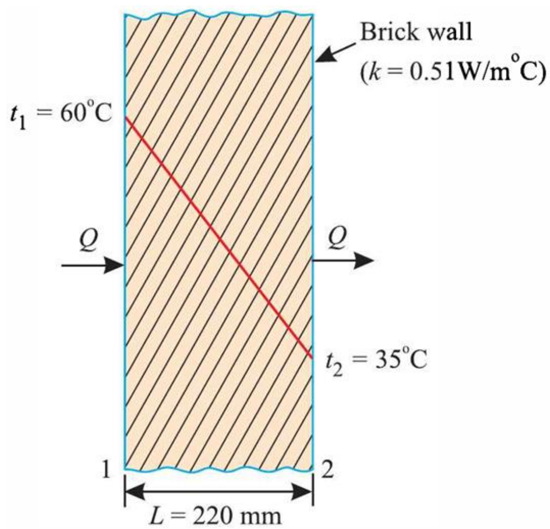


Fig. 2.10.

Example 2.3. *Consider a slab of thickness L = 0.25 m. One surface is kept at 100°C and the other surface at 0°C. Determine the net flux across the slab if the slab is made from pure copper. Thermal conductivity of copper may be taken as 387.6 W/m K.*

(AMIE Winter, 1998)

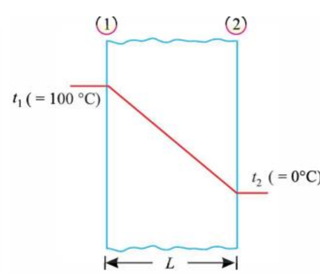


Fig. 2.11.

Solution. *Given* : $L = 0.25$ m; $t_1 = 100°C$; $t_2 = 0°C$; $k = 387.6$ W/m K.

From Fourier's law,

$$Q = -kA\frac{dt}{dx} \quad ...[\text{Eqn. } (1.1)]$$

Net flux,

$$q = \frac{Q}{A} = -k \cdot \frac{(t_2 - t_1)}{L}$$

$$= -387.6 \times \frac{(0 - 100)}{0.25}$$

$$\mathbf{= 1.55 \times 10^5\ W/m^2 \quad (Ans.)}$$

Example 2.4. *A reactor's wall, 320 mm thick, is made up of an inner layer of fire brick (k = 0.84 W/m°C) covered with a layer of insulation (k = 0.16 W/m°C). The reactor operates at a temperature of 1325°C and the ambient temperature is 25°C.*

(i) Determine the thickness of fire brick and insulation which gives minimum heat loss;

(ii) Calculate the heat loss presuming that the insulating material has a maximum temperature of 1200°C.

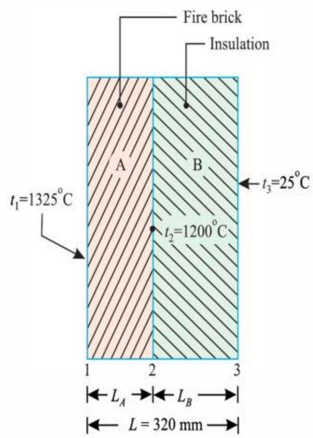


Fig. 2.12.

If the calculated heat loss is not acceptable, then state whether addition of another layer of insulation would provide a satisfactory solution.

Solution. Refer to Fig. 2.12.

Given : $t_1 = 1325°C;\ t_2 = 1200°C,\ t_3 = 25°C$

$$L_A + L_B = L = 320 \text{ mm or } 0.32 \text{ m}$$

$$\therefore \quad L_B = (0.32 - L_A); \qquad ...(i)$$

$$k_A = 0.84 \text{ W/m°C};$$

$$k_B = 0.16 \text{ W/m°C}.$$

(*i*) $L_A : L_B$:

The heat flux, under steady state conditions, is constant throughout the wall and is same for each layer. Then for *unit area* of wall,

$$q = \frac{t_1 - t_3}{L_A / k_A + L_B / k_B} = \frac{t_1 - t_2}{L_A / k_A} = \frac{t_2 - t_3}{L_B / k_B}$$

Considering first two quantities, we have

$$\frac{(1325 - 25)}{L_A / 0.84 + L_B / 0.16} = \frac{(1325 - 1200)}{L_A / 0.84}$$

or, $$\frac{1300}{1.190\, L_A + 6.25\,(0.32 - L_A)} = \frac{105}{L_A}$$

or, $$\frac{1300}{1.190\, L_A + 2 - 6.25\, L_A} = \frac{105}{L_A}$$

or, $$\frac{1300}{2 - 5.06\, L_A} = \frac{105}{L_A}$$

or, $$1300\, L_A = 105\,(2 - 5.06\, L_A)$$

or, $$1300\, L_A = 210 - 531.3\, L_A$$

or, $$L_A = \frac{210}{(1300 + 531.3)} = 0.1146 \text{ m or } \mathbf{114.6\ mm}$$

$\therefore$ Thickness of insulation $L_B = 320 - 114.6 = \mathbf{205.4\ mm\ (Ans.)}$

(*ii*) **Heat loss per unit area, *q* :**

Heat loss per unit area, $$q = \frac{t_1 - t_2}{L_A / k_A} = \frac{1325 - 1200}{0.1146/0.84} = \mathbf{916.23\ W/m^2\ (Ans.)}$$

If another layer of insulating material is added, the heat loss from the wall will reduce; consequently the temperature drop across the fire brick lining will drop and the interface temperature t_2 will rise. As the interface temperature is *already fixed*, therefore, a *satisfactory solution will not be available by adding another layer of insulation.*

Example 2.5. *A wall of a furnace is made up of inside layer of silica brick 120 mm thick covered with a layer of magnesite brick 240 mm thick. The temperatures at the inside surface of silica brick wall and outside surface of magnesite brick wall are 725°C and 110°C respectively. The contact thermal resistance between the two walls at the interface is 0.0035°C/W per unit wall area. If thermal conductivities of silica and magnesite bricks are 1.7 W/m°C and 5.8 W/m°C, calculate.*

(*i*) *The rate of heat loss per unit area of walls, and*

(*ii*) *The temperature drop at the interface.*

Solution. Refer Fig. 2.13.

Given : $L_A = 120$ mm $= 0.12$ m;

$L_B = 240$ mm $= 0.24$ m;

$k_A = 1.7$ W/m°C; $k_B = 5.8$ W/m°C

The contact thermal resistance $(R_{th})_{cont.}$ $= 0.0035$°C/W

The temperature at the inside surface of silica brick wall, $t_1 = 725$°C

The temperature at the outside surface of the magnesite brick wall, $t_4 = 110$°C

(*i*) **The rate of heat loss per unit area of wall, *q* :**

$$q = \frac{\Delta t}{\Sigma R_{th}} = \frac{\Delta t}{R_{th-A} + (R_{th})_{cont.} + R_{th-B}}$$

$$= \frac{(t_1 - t_4)}{L_A / k_A + 0.0035 + L_B / k_B}$$

$$= \frac{(725 - 110)}{0.12/1.7 + 0.0035 + 0.24/5.8}$$

$$= \frac{615}{0.0706 + 0.0035 + 0.0414}$$

$$= 5324.67\ \text{W/m}^2$$

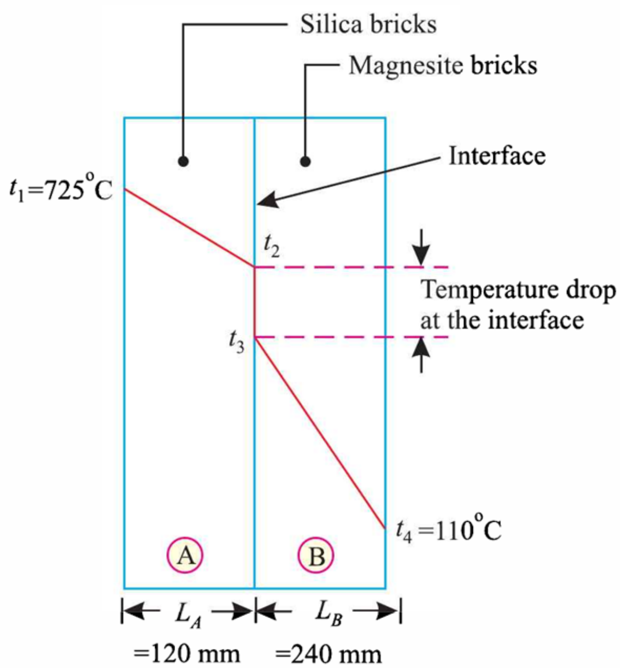


Fig. 2.13.

∴ The rate of heat loss per unit area of wall, q = **5324.67 W/m²** **(Ans.)**

(ii) The temperature drop at the interface, $(t_2 - t_3)$:

As the same heat flows through each layer of composite wall, therefore,

$$q = \frac{t_1 - t_2}{L_A / k_A} = \frac{t_3 - t_4}{L_B / k_B}$$

or, $$5324.67 = \frac{(725 - t_2)}{0.12/1.7}$$

or, $$t_2 = 725 - 5324.67 \times \frac{0.12}{1.7} = 349.14°C$$

Similarly, $$5324.67 = \frac{(t_3 - 110)}{0.24/5.8}$$

or, $$t_3 = 110 + 5324.67 \times \frac{0.24}{5.8} = 330.33°C$$

Hence, the temperature drop at the interface = $t_2 - t_3$

= 349.14 – 330.33 = **18.81°C** **(Ans.)**

Example 2.6. *An exterior wall of a house may be approximated by a 0.1 m layer of common brick (k = 0.7 W/m°C) followed by a 0.04m layer of gypsum plaster (k = 0.48 W/m°C). What thickness of loosely packed rock wool insulation (k = 0.065 W/m°C) should be added to reduce the heat loss or (gain) through the wall by 80 per cent ?* **(AMIE Summer, 1999)**

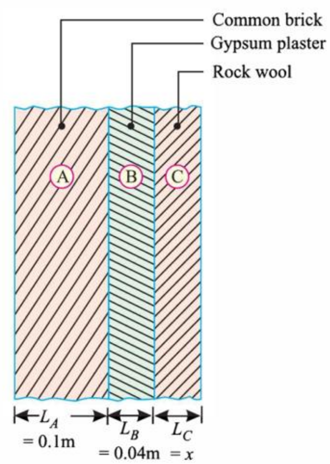


Fig. 2.14.

Solution. Refer to Fig. 2.14.

Thickness of common brick, L_A = 0.1 m

Thickness of gypsum plaster, L_B = 0.04 m

Thickness of rock wool, L_C = x (in m) = ?

Thermal conductivities :

Common brick, k_A = 0.7 W/m°C;

Gypsum plaster, k_B = 0.48 W/m°C;

Rock wool, k_C = 0.065 W/m°C.

Case I. *Rock wool insulation not used* :

$$Q_1 = \frac{A(\Delta t)}{\frac{L_A}{k_A} + \frac{L_B}{k_B}} = \frac{A(\Delta t)}{\frac{0.1}{0.7} + \frac{0.04}{0.48}} \quad ...(i)$$

Case II. *Rock wool insulation used* :

$$Q_2 = \frac{A(\Delta t)}{\frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C}} = \frac{A(\Delta t)}{\frac{0.1}{0.7} + \frac{0.04}{0.48} + \frac{x}{0.065}} \quad ...(ii)$$

But, $Q_2 = (1 - 0.8)\, Q_1 = 0.2\, Q_1$ ...(*given*)

$$\therefore \quad \frac{A(\Delta t)}{\frac{0.1}{0.7} + \frac{0.04}{0.48} + \frac{x}{0.065}} = 0.2 \times \frac{A(\Delta t)}{\frac{0.1}{0.7} + \frac{0.04}{0.48}}$$

or, $$\frac{0.1}{0.7} + \frac{0.04}{0.48} = 0.2\left[\frac{0.1}{0.7} + \frac{0.04}{0.48} + \frac{x}{0.065}\right]$$

or, $$0.1428 + 0.0833 = 0.2\,[0.1428 + 0.0833 + 15.385x]$$

or, $$0.2261 = 0.2\,(0.2261 + 15.385\,x)$$

or, $$x = 0.0588 \text{ m or } 58.8 \text{ mm}$$

Thus, the *thickness of rock wool insulation should be* **58.8 mm** **(Ans.)**

Example 2.7. *A furnace wall consists of 200 mm layer of refractory bricks, 6 mm layer of steel plate and a 100 mm layer of insulation bricks. The maximum temperature of the wall is 1150°C on the furnace side and the minimum temperature is 40°C on the outermost side of the wall. An accurate energy balance over the furnace shows that the heat loss from the wall is 400 W/m². It is known that there is a thin layer of air between the layers of refractory bricks and steel plate. Thermal conductivities for the three layers are 1.52, 45 and 0.138 W/m°C respectively. Find :*

(*i*) *To how many millimeters of insulation brick is the air layer equivalent?*

(*ii*) *What is the temperature of the outer surface of the steel plate?*

(AMIE Winter, 1996)

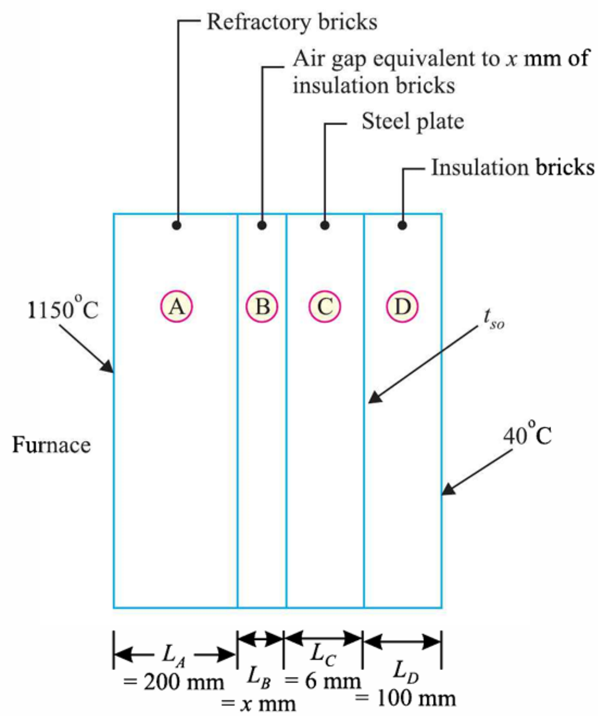


Fig. 2.15.

Solution. Refer Fig. 2.15.

Thickness of refractory bricks,

L_A = 200 mm = 0.2 m