Determinants :- المحددات

square array of numbers or symbles and closed by two vertical bars.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}|$$

a_{ij} = individual elements.
i = row
j = column
n = order of determinant
D = (always be) value of determinant.

Miner a_{ij} :-

Designated as M_{ij} of order $n-1$

$$a_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \Rightarrow M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

ويعني المصفوفة المتبقية بعد الحذف تدعى Miner.

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CoFactor of a_{ij} :-

Designated by A_{ij} is defined by :-

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Laplace Expansion Formula :-

$$\text{Det } A = \sum_{j=1}^{n} a_{ij} A_{ij} \leftarrow (i = \text{constant, summing along row } i)$$

$$\text{Det } A = \sum_{i=1}^{n} a_{ij} A_{ij} \leftarrow (\text{summing along column } j)$$

Example

Find the Determine of $D = \begin{vmatrix} 3 & 0 & -2 \\ 1 & 4 & 1 \\ 3 & -2 & 5 \end{vmatrix}$

Solution

- Lets do it about first column, which means $j = 1$ = constant

$$\therefore D = (3)(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} + (1)(-1)^{1+2}\begin{vmatrix} 0 & -2 \\ -2 & 5 \end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix}$$

$\therefore D = 94$

- Lets expansion it about 1st row, which means $i = 1$ = Constant

$$\therefore D = (3)(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} + (0)(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + (-2)(-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}$$

$= 94$ ok

Matrices :- المصفوفات

A rectangular array of numbers or symbols with each element being distinct and seperate (m rows, n columns) is called an m×n matrix when certain laws of combination, yet to be specified, are laid down,

Example :-

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{where } i = 1, 2, \dots, m$$

Solution

$$y_1 = a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n$$

$$\vdots$$

$$y_m = a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n$$

This can be written as a matrix

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{[Y]} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{[X]}$$

$$[Y] = [A][X]$$

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Special Matrices :-

① **Square matrix :-**

$$[A] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{nn} \end{bmatrix} \longrightarrow (m = n)$$

② **Diagonal matrix :-**

$$[A] = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & a_{33} & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

③ **Unit matrix :-** It may called by Identity matrix

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

④ **Upper triangular matrix :-**

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

⑤ **Upper unit triangular matrix :-**

$[T] = [U]$ with diagonal elements equal to 1.

(6) Lower triangular matrix :-

[L] = all elements above the diagonal are zero

(7) Inverse of Square matrix [A] :-

Defined by:-

(Identity or unit matrix)

$$[A]^{-1}[A] = [I] \qquad a^{-1}a = \frac{\cancel{a}}{\cancel{a}} = 1$$

where,

$[A]^{-1}$ is called A inverse

(8) Tridiagonal matrix :- مصفوفة قطرية

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & \text{Zero} & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & \text{Zero} & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

(9) Transpose of a matrix :-

A matrix Form when Columns & rows are interchange, designated as A^T or $[A]^T$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow [A]^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

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(10) Elemantry matrix operations :-

a- Two matrices are equal

$[A] = [B]$ if $a_{ij} = b_{ij}$, $\begin{cases} i = 1,2,---,m \\ j = 1,2,---,n \end{cases}$

b- Matrix addition:-

$[A] + [B] = [C]$ if $c_{ij} = a_{ij} + b_{ij}$

c- Matrix multiplication :-

$[C] = [A][B]$ if $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$\begin{cases} i = 1,2,---,m \\ k = 1,2,---,q \end{cases}$

Example :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} \overset{c_{11}}{(a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})} & \overset{c_{12}}{(a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})} \\ \underset{c_{21}}{(a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})} & \underset{c_{22}}{(a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Matrix properties :-

1- Matrix multiplication is associative

$[A]\{[B][C]\} = \{[A][B]\}[C]$

2- Not Commutative

$[A][B] \neq [B][A]$

But there is exception $\Rightarrow [I][A] = [A][I] = [A]$

3- Is distributive $\Rightarrow [A]\{[B] + [C]\} = [A][B] + [A][C]$

Properties of Determinant :-

a- $\text{Det}[I] = 1$

b- $\text{Det}[A] = \text{Det}[A]^{-1}$

c- $(\text{Det}[A])^{-1} = \dfrac{1}{\text{Det}[A]}$

d- If $[A]$ and $[B]$ are square matrix, then

$$\text{Det}([A][B]) = \text{Det}[A] \cdot \text{Det}[B]$$

e- If $[A]$ has size $n \times n$ and K is a constant, then,

$$\text{Det}(K[A]) = K^n \cdot \text{Det}[A]$$

f- If to the element of any row (or column) are added K times the corresponding elements of any other row (or column), the determinant is unchanged

Example

Find determinant

$$\begin{array}{c} \quad \times -2 \\ D = \begin{vmatrix} 3 & 1 & 3 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix} = -8 \end{array}$$

And, with property "f":

$$D = \begin{vmatrix} 3 & 1 & (3-\overset{1}{(2*1)}) \\ 0 & 2 & (4-\overset{0}{(2*2)}) \\ 1 & 2 & (3-\overset{-1}{(2*2)}) \end{vmatrix} = -8$$

g- If any two rows (or columns) of $[A]$ are interchanged, yielding a new matrix $[B]$, then

$$\text{Det}[B] = -\text{Det}[A]$$

h- If [A] is triangular, then Det[A] is the product of the diagonal elements.

$$\text{Det}[A] = |A| = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

Example: Let

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix}$$

Find the Det[A] with several different methods.

Solution:

① By using "Laplace Expansion formula":

$$\text{Det } A = \sum_{j=1}^{n} a_{ij} A_{ij} \quad ; \quad A_{ij} = (-1)^{i+j} M_{ij}$$

a- Using row #1

$$\text{Det } A = |A| = (0)\begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} - (2)\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - (2)[-16 - 10] + (-1)[0 - 6]$$

$$|A| = \boxed{58}$$

b- Using row #2:

$$\text{Det } A = |A| = -(4)\begin{vmatrix} 2 & -1 \\ 0 & -4 \end{vmatrix} + (3)\begin{vmatrix} 0 & -1 \\ 2 & -4 \end{vmatrix} - (5)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -(4)(-8) + (3)(2) - (5)(-4)$$

$$|A| = \boxed{58}$$

c- Using property (g), then (f) to get "Upper triangular matrix"

$$|A| = \begin{vmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & -4 \\ 4 & 3 & 5 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 2 & -4 \end{vmatrix} \Leftarrow \{a_{2j} - 2(a_{1j})\}$$

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$$= -\begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 0 & -\frac{29}{3} \end{vmatrix} \leftarrow \{a_{3j} - \tfrac{2}{3}(a_{2j})\}$$

Now we have "upper triangular matrix"

$$= -(2)(3)\left(-\frac{29}{3}\right)$$

$$= \boxed{58}$$

i- If all the elements of any row or column are zero, then Det $A = 0$

j- If any one row (or column) a of A is separated as $a = b + c$, then

$$\text{Det } A|_a = \text{Det } A|_b + \text{Det } A|_c$$

Ex)

$$\begin{vmatrix} 6+2 & -3+1 & 5+4 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix} = \begin{vmatrix} 6 & -3 & 5 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 6 & 7 \end{vmatrix}$$

k- The determinant of A & its transpose are equal

$$\text{Det } A^T = \text{Det } A$$

l- In general $\text{Det}(A+B) \neq \text{Det } A + \text{Det } B$

Cramer's Rule : {The name is coming after Gabriel Cramer (1704 – 1752)}

$$x_i = \frac{\text{Det } A_i}{\text{Det } A} \quad ; \quad [A][X] = [B]$$

where;

Det A = Determinant of matrix [A].

Det A_i = Determinant of matrix $[A_i]$, where $[A_i]$ is matrix [A] with its i-th column replaced by column matrix [B].

Ex)

$$3x_1 - 3x_2 + 5x_3 = 4$$
$$x_1 + 2x_2 - 6x_3 = 3$$
$$2x_1 - x_2 + 3x_3 = 1$$

Find $x_1, x_2, \& \ x_3$.

Solution

In the matrix form, we have:

$$\underbrace{\begin{bmatrix} 3 & -3 & 5 \\ 1 & 2 & -6 \\ 2 & -1 & 3 \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{[x]} = \underbrace{\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}}_{[B]}$$

We can find Det A = 20

$$\text{Det } A_1 = \begin{vmatrix} 4 & -3 & 5 \\ 3 & 2 & -6 \\ 1 & -1 & 3 \end{vmatrix} = 20$$

similarly,

$$\text{Det } A_2 = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 3 & -6 \\ 2 & 1 & 3 \end{vmatrix} = -40$$

&

Det A_3 = -20

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Then,

$$x_1 = \frac{20}{20} = 1 \ , \ x_2 = \frac{-40}{20} = -2 \ , \ x_3 = \frac{-20}{20} = -1$$

Application of the Determinant :-

المساعد * First we need to define the adjugate of (A) to be the transpose of the Cofactor matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{then,}} adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (A_{ij}^{T})$$

(Cofactor Transpose)

Inverse Matrix :-

$$A^{-1} = \frac{adj(A)}{Det(A)} \ , \quad Det(A) \neq 0$$

EX) Find the inverse of the matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, using the adjugate theorm.

Solution,

* First, we will find the determinant of the matrix

$$Det\ A = (1)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= -2 - 2(3-1) + 3(6) = -2 - 4 + 18$$

$$Det\ A = \boxed{12}$$

Now, we need to find adj(A). To do so, 1st we will find the cofactor matrix of (A). This is given by

$$\operatorname{Cof}(A) = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix} = A_{ij}$$

$$\therefore \operatorname{adj}(A) = A_{ij}^{T} = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{12}\begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

To verify our answer for (A^{-1}), compute the product $A \cdot A^{-1}$ &/or $A^{-1} \cdot A$ & make sure each product is equal to I (identity matrix)

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

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H.W

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

Using the "adjugate theorem"

-- نهاية محاضرة " Matrix Operations, Linear System Equations,
Solving Linear Systemby Matrix Methods عمليات المصفوفات، معكوس
المصفوفة، تمثيل انظمة المعادلات الخطية، حل انظمة المعادلات الخطة بطرق المصفوفات"--

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