



Multiple Integrals

The multiple integrals are the integrals of a function of two or more variables over a region in the plane or space.

Double Integrals :-

① If a function $f(x,y)$ is defined on a rectangular region R

$R: a \leq x \leq b \quad \& \quad c \leq y \leq d$, then

$$\iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$$

② If a function $f(x,y)$ is defined on a region R

$R: a \leq x \leq b \quad \& \quad f_1(x) \leq y \leq f_2(x)$, then

$$\iint_R f(x,y) dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) dy dx$$

③ If a function $f(x,y)$ is defined on a region R

$R: g_1(x) \leq x \leq g_2(x) \quad \& \quad c \leq y \leq d$

$$\iint_R f(x,y) dA = \int_c^d \int_{g_1(x)}^{g_2(x)} f(x,y) dx dy$$



Ex: Calculate $\iint_R f(x,y) dA$ for

$$f(x,y) = 1 - 6x^2y \quad \text{and} \quad R: 0 \leq x \leq 2, -1 \leq y \leq 1$$

Solution:

$$\begin{aligned} \iint_R f(x,y) dA &= \int_0^2 \int_{-1}^1 (1 - 6x^2y) dy dx = \int_0^2 \left[y - \frac{6x^2y^2}{2} \right]_{-1}^1 dx \\ &= \int_0^2 \left[(1 - \frac{6x^2 \cdot 1^2}{2}) - (-1 - \frac{6x^2 \cdot (-1)^2}{2}) \right] dx \\ &= \int_0^2 [1 - 3x^2 - (-1 - 3x^2)] dx = \int_0^2 2 dx = 2x \Big|_0^2 \\ &= (2 \cdot 2) - (2 \cdot 0) = \boxed{4} \end{aligned}$$

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Ex: Calculate $\iint_0^1 e^{y^2} dx dy$.

HW next week

Solution:

$$\begin{aligned} \iint_0^1 e^{y^2} dx dy &= \int_0^1 e^{y^2} \cdot x \Big|_0^y dy = \int_0^1 [(e^{y^2} \cdot y) - (e^{y^2} \cdot 0)] dy \\ &= \int_0^1 y e^{y^2} dy \cdot \frac{1}{2} = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2} (e^1 - e^0) \\ &= \boxed{\frac{1}{2} (e^1 - 1)} \end{aligned}$$

Ex: Find $\iint_0^1 \int_{y^2}^{6-y} dx dy$

Solution:

$$\begin{aligned} \iint_0^1 \int_{y^2}^{6-y} dx dy &= \int_0^1 x \Big|_{y^2}^{6-y} dy = \int_0^1 (6-y - y^2) dy = 6y - \frac{y^2}{2} - \frac{y^3}{3} \Big|_0^1 \\ &= (6 \cdot 2 - \frac{2^2}{2} - \frac{2^3}{3}) - (0) = 12 - 2 - \frac{8}{3} \\ &= \frac{22}{3} = \boxed{7.33} \end{aligned}$$



Ex:- Calculate $\iint_R f(x,y) dA$ for Quiz_17-11-2024

$$f(x,y) = 4x+2 \quad \& \quad R: x^2 \leq y \leq 2x, 0 \leq x \leq 2$$

Solutions:-

$$\iint_R f(x,y) dA = \int_0^2 \int_{x^2}^{2x} (4x+2) dy dx = \int_0^2 [(4xy+2y)]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 [(4x \cdot 2x + 2 \cdot 2x) - (4x \cdot x^2 + 2 \cdot x^2)] dx$$

$$= \int_0^2 [8x^2 + 4x - (4x^3 + 2x^2)] dx = \int_0^2 (6x^2 + 4x - 4x^3) dx$$

$$= \left[\frac{6x^3}{3} + \frac{4x^2}{2} - \frac{4x^4}{4} \right]_0^2 = [2x^3 + 2x^2 - x^4]_0^2$$

$$= 2(2^3) + 2(2)^2 - (2)^4 = 16 + 8 - 16 = \boxed{8}$$

Ex:- Evaluate the $\iint_R \frac{\sin x}{x} dA$ where R is the triangle in XY-Plane bounded by the x-axis the line $y=x$ and the line $x=1$.

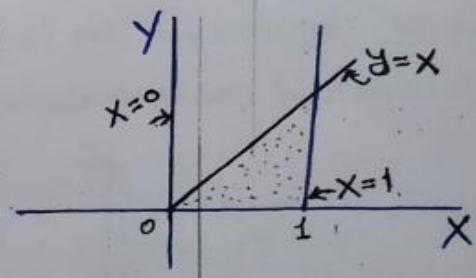
Solution:-

$$\iint_R \frac{\sin x}{x} dy dx = \int_0^1 \frac{\sin x}{x} \int_0^x dy dx$$

$$\int_0^1 \frac{\sin x}{x} \cdot x dx = [-\cos x]_0^1$$

$$= -(\cos(1) - \cos(0))$$

$$= \boxed{0.459}$$



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Determining the Limits of Integration:-

If we want to evaluate $\iint f(x,y) dA$ over the region R shown in Fig.(1), integrating first with respect to (y) and then with respect to (x), we take the following steps:

① we imagine a vertical line L cutting through R in the direction of increasing (y) as shown in Fig.(2).

② We integrate from the y-value where L enters R to the y-value where L leaves R.

③ We choose x-limits that include all the vertical lines that pass through R.

$$\therefore \iint_R f(x,y) dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y) dy dx$$

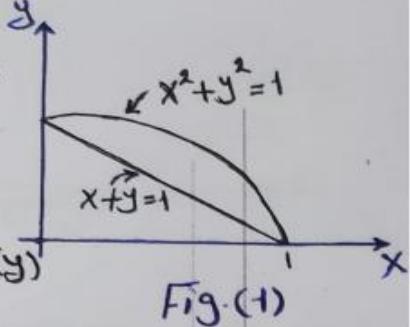


Fig.(1)

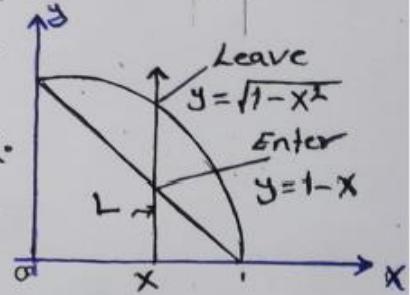


Fig.(2)

To calculate the same double integral as an integral with the order of integration reversed the procedure uses horizontal line as shown in Fig.(3)

$$\therefore \iint_R f(x,y) dA = \int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y) dx dy.$$

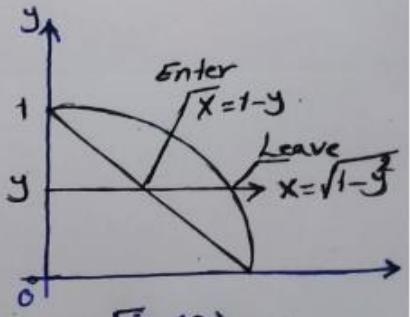


Fig.(3)

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Ex:- Find an equivalent integral to the $\iint_{0 \leq x^2 \leq 2x} (4x+2) dy dx$ by the order of the integral reversed.

Solution :-

$$R: x=0 \text{ to } x=2$$

$$y=x^2 \text{ to } y=2x$$

$$\therefore \iint_{0 \leq x^2 \leq 2x} (4x+2) dy dx = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2) dx dy$$

$$\int_{y/2}^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2) dx dy = \int_0^4 \left[\frac{4x^2}{2} + 2x \right]_{y/2}^{\sqrt{y}} dy$$
$$= \int_0^4 \left[(2(\sqrt{y})^2 + 2\sqrt{y}) - (2(\frac{y}{2})^2 + 2 \cdot \frac{y}{2}) \right] dy$$

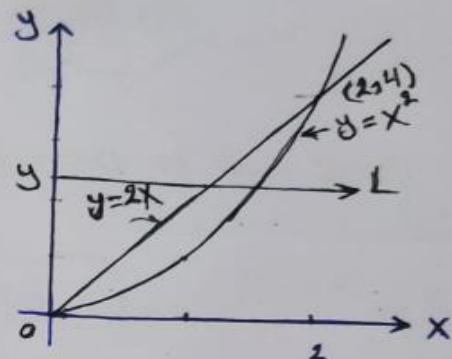
$$= \int_0^4 \left(2y + 2\sqrt{y} - \frac{y^2}{2} - y \right) dy = \int_0^4 \left(y + 2\sqrt{y} - \frac{y^2}{2} \right) dy$$

$$= \left[\frac{y^2}{2} + \frac{2(y)^{3/2}}{3/2} - \frac{y^3}{2 \cdot 3} \right]_0^4$$

$$= \left(\frac{4^2}{2} + \frac{4}{3} (4)^{3/2} - \frac{(4)^3}{6} \right)$$

$$= 8 + \frac{4}{3}(8) - \frac{32}{3}$$

$$= \boxed{8}$$





Areas, Moments and Center of Mass :-

The area of a closed bounded plane region R is:

$$A = \iint_R dA = \iint_R dx dy = \iint_R dy dx$$

To find the moments and Center of mass of thin sheet or plate we use the formulas :-

① Mass :- $M = \iint \delta(x, y) \cdot dA$

where $\delta(x, y)$ is the density

② First moments :- $M_x = \iint y \cdot \delta(x, y) \cdot dA$

$$M_y = \iint x \cdot \delta(x, y) \cdot dA$$

③ Center of mass :-

$$\bar{x} = M_y / M$$

$$\bar{y} = M_x / M$$

④ Moment of inertia :-

- About x -axis

$$I_x = \iint y^2 \cdot \delta(x, y) \cdot dA$$

- About y -axis

$$I_y = \iint x^2 \cdot \delta(x, y) \cdot dA$$

- About the origin

$$I_o = \iint (x^2 + y^2) \cdot \delta(x, y) \cdot dA$$

⑤ Radii of gyration :-

- About x -axis

$$R_x = \sqrt{I_x / M}$$

- About y -axis

$$R_y = \sqrt{I_y / M}$$

- About the origin

$$R_o = \sqrt{I_o / M}$$



Quiz 26-11-2024

Ex:- A thin plate covers the plane region R bounded by $y = x^2$, $y = \sqrt{2-x^2}$ and y -axis and with a density function $\delta = xy$

- Find the mass of the body
- Find the center of the mass
- Find the radii of gyration

Solution:-

$$x^2 = \sqrt{2-x^2} \Rightarrow x^4 = 2-x^2$$

$$x^4 + x^2 - 2 = 0$$

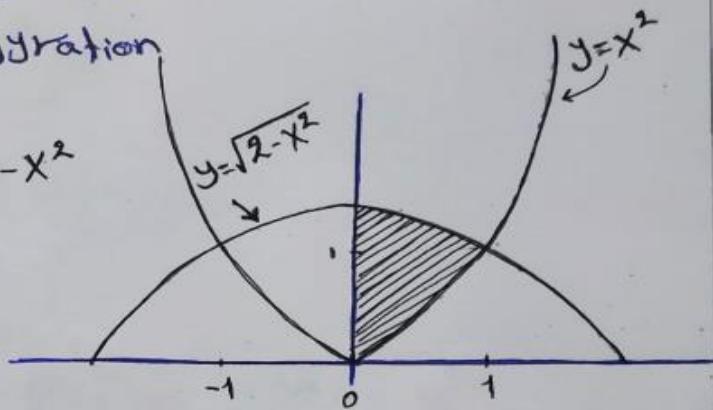
$$(x^2+2)(x^2-1) = 0$$

$$x^2-1 = 0 \Rightarrow x^2 = 1$$

$$\therefore x = \pm 1$$

$$x^2+2=0 \Rightarrow x^2 = -2 \quad \text{dotted}$$

$$\therefore y = 1^2 = 1$$



$$\begin{aligned} a) M &= \iint_R \delta(x,y) dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} xy \cdot dy dx \\ &= \int_0^1 x \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{2-x^2}} dx = \int_0^1 x \left[\left(\frac{2-x^2}{2} \right) - \frac{(x^2)^2}{2} \right] dx \\ &= \int_0^1 x \left[\frac{2-x^2}{2} - \frac{x^4}{2} \right] dx \\ &= \int_0^1 \left[x - \frac{x^3}{2} - \frac{x^5}{2} \right] dx = \left[\frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{12} \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{8} - \frac{1}{12} = \boxed{\frac{7}{24}} \end{aligned}$$

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$$\begin{aligned} b) M_x &= \iint_R y \cdot g(x, y) \cdot dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x \cdot y^2 \cdot dy \cdot dx \\ &= \int_0^1 x \cdot \frac{y^3}{3} \Big|_{x^2}^{\sqrt{2-x^2}} \cdot dx \\ &= \int_0^1 x \cdot \left[\left(\frac{(\sqrt{2-x^2})^3}{3} \right) - \left(\frac{(x^2)^3}{3} \right) \right] \cdot dx \\ &= \int_0^1 \left[\frac{x(\sqrt{2-x^2})^{3/2}}{3} - \frac{x^7}{3} \right] \cdot dx \\ &= \left[\frac{-1}{2 \cdot 3} \cdot \frac{(2-x^2)^{5/2}}{5/2} - \frac{x^8}{8 \cdot 3} \right]_0^1 = \\ &= \left[\left(\frac{-1}{15} (2-1)^{5/2} - \frac{1^8}{24} \right) \right] - \left[\left(\frac{-1}{15} (2-0^2)^{5/2} \right) - \frac{0^8}{24} \right] \\ &= \boxed{0.268} \end{aligned}$$

$$\therefore \bar{y} = M_x / M = \frac{0.268}{7/24} = \boxed{0.922}$$

$$\begin{aligned} M_y &= \iint_R x \cdot g(x, y) \cdot dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^2 y \cdot dy \cdot dx \\ &= \int_0^1 x^2 \cdot \frac{y^2}{2} \Big|_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x^2 \left[\frac{(\sqrt{2-x^2})^2}{2} - \frac{(x^2)^2}{2} \right] \cdot dx \\ &= \int_0^1 x^2 \left[\frac{2-x^2}{2} - \frac{x^4}{2} \right] \cdot dx = \int_0^1 \left(x^2 - \frac{x^4}{2} - \frac{x^6}{2} \right) \cdot dx \\ &= \left[\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{14} \right]_0^1 = \frac{1}{3} - \frac{1}{10} - \frac{1}{14} = \boxed{\frac{17}{105}} \\ \therefore \bar{x} &= M_y / M = \frac{17/105}{7/24} = \boxed{0.56} \end{aligned}$$



$$\begin{aligned} C/ I_x &= \iint_R y^2 \cdot \delta(x,y) dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x y^3 \cdot dy \cdot dx \\ &= \int_0^1 x \left[\frac{y^4}{4} \right]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x \left[\left(\sqrt{2-x^2} \right)^4 - (x^2)^4 \right] dx \\ &= \int_0^1 \left[\frac{x}{4} (2-x^2)^2 - \frac{x^9}{4} \right] dx = \left[\frac{-1}{4 \cdot 2} \frac{(2-x^2)^3}{3} - \frac{x^{10}}{40} \right]_0^1 \\ &= \left[\frac{-1}{24} (2-1)^3 - \frac{1}{40} \right] - \left[\frac{-1}{24} (2-0)^3 - \frac{0}{40} \right] \\ &= \frac{-1}{24} - \frac{1}{40} + \frac{8}{24} = \frac{4}{15} = 0.27 \end{aligned}$$

$$\therefore R_x = \sqrt{I_x/M} = \sqrt{\frac{4/15}{7/24}} = 0.96$$

$$\begin{aligned} I_y &= \iint_R x^2 \cdot \delta(x,y) dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^3 y \cdot dy \cdot dx \\ &= \int_0^1 x^3 \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x^3}{2} \left[(\sqrt{2-x^2})^2 - (x^2)^2 \right] dx \\ &= \int_0^1 \left[x^3 - \frac{x^5}{2} - \frac{x^7}{2} \right] dx = \left[\frac{x^4}{4} - \frac{x^6}{12} - \frac{x^8}{16} \right]_0^1 \\ &= \frac{1}{4} - \frac{1}{12} - \frac{1}{16} = \frac{5}{48} \end{aligned}$$

$$\therefore R_y = \sqrt{I_y/M} = \sqrt{\frac{5/48}{7/24}} = 0.597$$

$$I_o = I_x + I_y = \frac{4}{15} + \frac{5}{48} = \frac{89}{240}$$

$$R_o = \sqrt{I_o/M} = \sqrt{\frac{89/240}{7/24}} = 1.13$$



Triple Integrals :-

The integral $\iiint_D f(x, y, z) dV$ is called the volume integral and it is evaluated in the manner that

$$\iiint_D f(x, y, z) \cdot dV = \iint_R \left(\int_{z_1}^{z_2} f(x, y, z) dz \right) \cdot dA$$

The volume of a closed bounded region D in Space is

$$V_D = \iiint_D dV$$

Ex:- Evaluate $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz \cdot dx \cdot dy$

$$\begin{aligned} \text{Solution:-} \quad & \int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz \cdot dx \cdot dy = \int_0^{\sqrt{2}} \int_0^{3y} [(8-x^2-y^2) - (x^2+3y^2)] \cdot dx \cdot dy \\ & = \int_0^{\sqrt{2}} \int_0^{3y} (8-2x^2-4y^2) \cdot dx \cdot dy = \int_0^{\sqrt{2}} \left[8x - \frac{2x^3}{3} - 4y^2 x \right]_0^{3y} \cdot dy \\ & = \int_0^{\sqrt{2}} \left[(8 \cdot 3y - \frac{2}{3} (3y)^3 - 4y^2 \cdot 3y) \right] dy = \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3) \cdot dy \\ & = \int_0^{\sqrt{2}} (24y - 30y^3) \cdot dy = \left[\frac{24y^2}{2} - \frac{30y^4}{4} \right]_0^{\sqrt{2}} \\ & = 12(\sqrt{2})^2 - \frac{15}{2} (\sqrt{2})^4 = 24 - 30 \\ & = \boxed{-6} \end{aligned}$$



H-W

① Evaluate the integral:

$$a - \int_0^2 \int_{-1}^1 (x-y) dy dx$$

$$b - \int_0^1 \int_1^2 x y e^x dy dx$$

$$c - \int_0^2 \int_1^{\ln 5} e^{2x+y} dy dx$$

② Evaluate the double integral over the region R

$$a/ \iint_R \frac{\sqrt{x}}{y^2} dA \quad R: 0 \leq x \leq 4 \quad \& \quad 1 \leq y \leq 2$$

$$b/ \iint_R x y \cos y dA \quad R: -1 \leq x \leq 1 \quad \& \quad 0 \leq y \leq \pi$$

$$c/ \iint_R \frac{xy^3}{x^2+1} dA \quad R: 0 \leq x < 1 \quad \& \quad 0 \leq y \leq 2$$

③ Sketch the region of integration and write an equivalent double integral with the order of integration reversed

$$a/ \int_0^1 \int_2^{4-2x} \cdot dy dx$$

$$b/ \int_0^1 \int_{1-x}^{1-x^2} \cdot dy dx$$

$$c/ \int_0^1 \int_y^{\sqrt{y}} \cdot dx dy$$

④ Sketch the region bounded by the parabolas $x=y^2$ and $x=2y-y^2$. Then express the region's area as an iterated double integral and evaluate the integral.

⑤ Evaluate $\int_0^{\ln 2} \int_0^x \cdot dy dx$



6- A thin plate cover the triangle region bounded by the x-axis and the line $x=1$ and $y=2x$ in the first quarter. The plate density is $\delta(x,y) = 6x+6y+6$. find

- a/ The mass of the body.
- b/ The center of mass.
- c/ The radii of gyration.

7- Evaluate

$$a/ \int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz \cdot dy \cdot dx$$

$$b/ \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz \cdot dy \cdot dx$$

$$c/ \int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \cdot dx \cdot dy \cdot dz$$



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