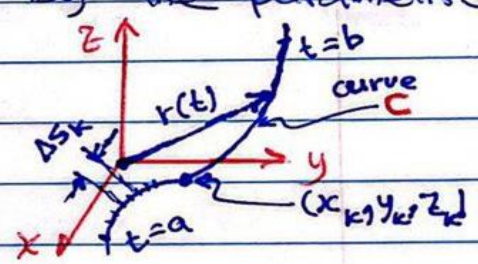
Line Integrals :-

In this topic, we are going to introduce a new kind of integral. Here we do integration over a curve C rather than over an interval [a, b].

Line integrals are used to find the work done by a force in moving an object along a path, & to find the mass of a curved wire with variable density.

If $f(x,y,z)$ is a real valued function that we need to integrate over the curve C that lying within its domain and this curve is parametrized by the parametric equations:

$x = g(t)$
$y = h(t)$
$z = k(t)$



where these parametric eqs can be expressed in vector valued function as:

$r(t) = g(t)\,i + h(t)\,j + k(t)\,k$

along the interval $a \le t \le b$.

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The values of f along the curve are given by the composite function $f(g(t), h(t), k(t))$.

To integrate this composite Fun with respect to arc length from $t=a$ to $t=b$, we 1st partition the curve C into a finite number n of subarcs that has length ΔS_K which has a point (x_k, y_k, z_k) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta S_k$$

But, if f is continuous & the functions g, h & k have continuous 1st derivatives, then these sum (Σ) approaches a limit as n increases & the lengths ΔS_k approaches zero.

line integral of f over C is :-

$$\int_C f(x,y,z)\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta S_k$$

$$\int_C f(x,y,z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |v(t)|\, dt$$

$$|v(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Summary:

How to Evaluate a Line Integral?

To integrate a Continuous function $f(x,y,z)$ over a curve C:

① Find a smooth parametrization of C.

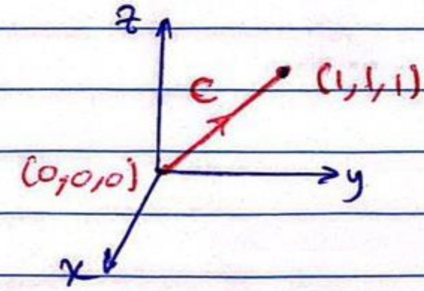
$$r(t) = g(t)\,i + h(t)\,j + K(t)\,K \quad , \quad a \le t \le b$$

② Evaluate the integral as;

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|V(t)|\,dt$$

Ex) Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1,1,1)$?

Sol)



* 1st we need to find the parametric equations in the form of vector valued function

From the vector lecture, the parametric eqs are;

$x = x_0 + At$, let $(x_0, y_0, z_0) = (0,0,0)$

$y = y_0 + Bt$ & $\vec{v} = i + j + K$

$z = z_0 + Ct$ $\therefore A = B = C = 1$

$\therefore \boxed{x = t \; ; \; y = t, \; z = t}$ ← Parametric eqs

$\therefore \vec{r}(t) = t\,i + t\,j + t\,K \qquad 0 \le t \le 1$

$\vec{v}(t) = \dfrac{d r(t)}{dt} = i + j + K$

$\therefore |\vec{v}(t)| = \sqrt{1^2+1^2+1^2} = \boxed{\sqrt{3}}$

$$\therefore \int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\,(\sqrt{3})\,dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt$$

$$= \sqrt{3}\int_0^1 (2t - 3t^2)\,dt$$

$$= \sqrt{3}\left[t^2 - t^3\right]_0^1 = \sqrt{3}\left[1^2 - 1^3 - (0)\right]$$

$$= \boxed{\text{zero}}$$ Ans

Notes

If a piecewise smooth curve C is made by joining a finite number of smooth curves $C_1, C_2, \ldots, C_n$ end to end, then the integral of a fun over C is the sum of the integrals over the curves that make it up:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds$$

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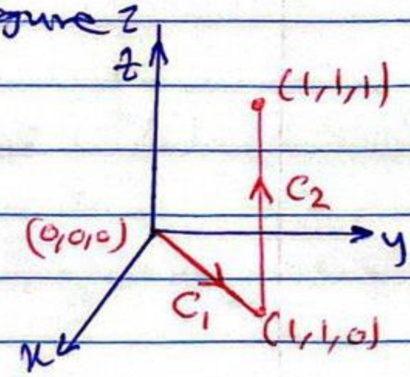
Ex) Integrate $f(x,y,z) = x - 3y^2 + z$ over $C_1 \cup C_2$ as shown in figure



sol.)

1st of all, we need to Find the parametric eqns of C_1 & C_2

① C_1 :- $0 \le t \le 1$

$\vec{v}_1 = (1-0)i + (1-0)j + (0-0)k = i + j$

$A_1 = B_1 = 1$, $C_1 = 0$

let $(x_0, y_0, z_0) = (0,0,0)$

$$\left.\begin{aligned} \therefore x_1 &= x_0 + A_1 t \Longrightarrow x = t \\ y_1 &= y_0 + B_1 t \Longrightarrow y = t \\ z_1 &= z_0 + C_1 t \Longrightarrow z = 0 \end{aligned}\right\} \Longrightarrow r(t) = ti + tj$$

$$v(t) = \frac{dr(t)}{dt} = i + j$$

$$|v(t)| = \sqrt{1^2 + 1^2} = \boxed{\sqrt{2}}$$

② C_2 :- $0 \le t \le 1$

$\vec{v}_2 = (1-1)i + (1-1)j + (1-0)k = k$

$A_2 = B_2 = 0$, $C_2 = 1$, let $(x_0, y_0, z_0) = (1,1,0)$

$$\left.\begin{aligned} \therefore x_2 &= x_0 + A_2 t \Longrightarrow x_2 = 1 \\ y_2 &= y_0 + B_2 t \Longrightarrow y_2 = 1 \\ z_2 &= z_0 + C_2 t \Longrightarrow z_2 = t \end{aligned}\right\} \Longrightarrow r(t) = i + j + tk$$

$$v(t) = \frac{dr(t)}{dt} = k$$

$$|v(t)| = \sqrt{0^2 + 0^2 + 1^2} = \boxed{1}$$

$$\therefore \int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$$= \int_0^1 f(t,t,0)\,|v(t)|\,dt + \int_0^1 f(1,1,t)\,|v(t)|\,ds$$

$$= \int_0^1 (t-3t^2+0)\sqrt{2}\,dt + \int_0^1 (1-3+t)(1)\,dt$$

$$= \sqrt{2}\left[\frac{t^2}{2}-t^3\right]_0^1 + \left[\frac{t^2}{2}-2t\right]_0^1$$

$$= \sqrt{2}\left[\frac{1}{2}-1\right] + \left[\frac{1}{2}-2\right]$$

$$= \sqrt{2}\left[-\frac{1}{2}\right] + \left[\frac{-3}{2}\right] = \boxed{-\frac{1}{\sqrt{2}} - \frac{3}{2}} \quad \text{Ans}$$

H.W :-

① Integrate $f(x,y,z) = \dfrac{x+y+z}{x^2+y^2+z^2}$ over the path $r(t) = t\,i + t\,j + t\,k$, on the interval $a \le t \le b$

② Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ over

$C_1 : r(t) = t\,k \qquad 0 \le t \le 1$

$C_2 : r(t) = t\,j + k \qquad 0 \le t \le 1$

$C_3 : r(t) = t\,i + j + k \qquad 0 \le t \le 1$

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③ Evaluate $\int_C (x+y)\,ds$ where C is the straight line segment $x=t$, $y=1-t$, $z=0$ from $(0,1,0)$ to $(1,0,0)$?

④ Find the line integral of $f(x,y,z)=x+y+z$ over the straight line segment from $(1,2,3)$ to $(0,-1,1)$?

⑤ Evaluate $\int_C (xy+y+z)\,ds$ along the curve $r(t)=2t\,i+t\,j+(2-2t)\,k$, $0\le t\le 1$?

⑥ Integrate $f(x,y,z)=x+\sqrt{y}-z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ over:

C_1 : $r(t)=t\,i+t^2\,j$, $0\le t\le 1$

C_2 : $r(t)=i+j+t\,k$, $0\le t\le 1$

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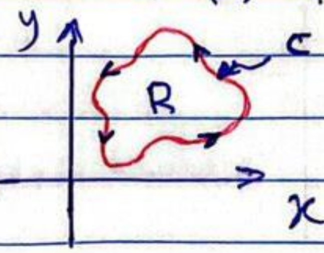
Green's Theorem :-

This theorem allows us to convert the line integral into double integral over the region R enclosed by C.

هنا شرط استخدام هذه النظرية هو وجود منطقة محاطة بحدود مغلقة والدوال التي تحت هذه المنطقة (region) يجب ان تكون مستمرة بجميع نقاط هذه المنطقة، ومعنى مستمرة هو ان المشتقة الاولى لهذه الدوال تكون موجودة في كل نقاط هذه المنطقة.



The orientation of the closed curve C is considerable.

Where the curve C has a positive orientation if it is traced out in a counter-clockwise (CCW) direction. & it has a negative orientation if it is traced out in a clockwise (CW) direction.

أي اتجاه حركة المنحني المغلق هو معتبر، حيث انه موجب اذا كان يعكس اتجاه عقارب الساعة +ve وسالب اذا كان باتجاه عقارب الساعة -ve.

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Subject (Math-3)
Lecturer (Dr Hussein K. Halwas)
1^{st} term – Lect. (Line Integrals and Green's Theorem)

Theorem: النظرية:

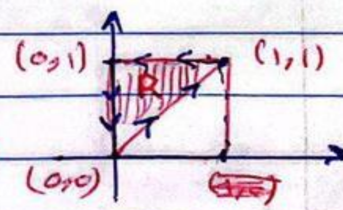
Let C be a positively oriented, piecewise smooth, simple, closed curve & the R be the region enclosed by the curve. If P & Q have continuous 1st order partial derivatives on R then,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Ex) Evaluate $\oint_C x\,dx - x^2y^2\,dy$, where C is the triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$ positively oriented?

Sol.)

بما أن المسار في التكامل اعلاه هي دوال صور مغلقة ومستمرة فهنا علينا تسهيل الامر على انفسنا باستخدام نظرية الخضراء وكذلك، وهنا يتحول التكامل الخطي الى تكامل ثنائي بسيط.



$$\oint_C \underbrace{\boxed{x}}_{P}\,dx \underbrace{\boxed{-x^2y^2}}_{Q}\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$P = x \Rightarrow \frac{\partial P}{\partial y} = \boxed{0}$$

$$Q = -x^2y^2 \longrightarrow \frac{\partial Q}{\partial x} = \boxed{-2xy^2}$$

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$$\oint_C x\,dx - x^2y^2\,dy = \int_0^1\int_x^1 (-2xy^2 - 0)\,dy\,dx$$

$$= \int_0^1 \left[\frac{-2}{3}xy^3\right]_x^1 dx = \int_0^1 \left[\frac{-2}{3}x - \left(\frac{-2}{3}x \cdot x^3\right)\right] dx$$

$$= \int_0^1 \left(\frac{-2x}{3} + \frac{2x^4}{3}\right) dx = \frac{-2}{3}\frac{x^2}{2} + \frac{2}{3}\frac{x^5}{5}\Big]_0^1$$

$$= \left(\frac{-1}{3} + \frac{2}{15}\right) - 0 = \frac{-1}{3} + \frac{2}{15} = \frac{-5+2}{15}$$

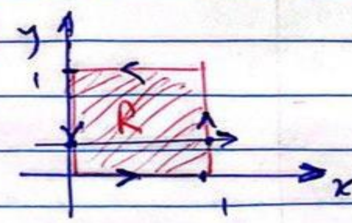
$$= \frac{-3}{15} = \boxed{\frac{-1}{5}} \quad \text{Ans}$$

Ex) Evaluate the line integral $\oint_C xy\,dy - y^2\,dx$, where C is the square cut from the 1st quadrant by the line $x = 1$ & $y = 1$

Sol.)

∵ التكامل خطي ومغلق فهنا يمكننا أن نستخدم النظرية "Green's theo."

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



$$\therefore P = -y^2 \longrightarrow \frac{\partial P}{\partial y} = -2y$$

$$Q = xy \longrightarrow \frac{\partial Q}{\partial x} = y$$

$$\therefore \oint_C xy\,dy - y^2\,dx = \int_0^1\int_0^1 (y + 2y)\,dx\,dy$$

$$= \int_0^1\int_0^1 3y\,dx\,dy = \int_0^1 3yx\Big]_0^1 dy$$

$$= \int_0^1 3y\,dy = \frac{3}{2}y^2\Big]_0^1 = \boxed{\frac{3}{2}} \quad \text{Ans}$$

----- نهاية محاضرة "التكاملات الخطية، النظرية الخضراء Line Intergrals, Green's Theorem

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