The Cylindrical Coordinate System (ρ, ø, z)

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry. A vector $\vec{A}$ in cylindrical coordinates can be written as:

$$\vec{A} = A_\rho \boldsymbol{a}_\rho + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_z$$

Where aρ , aø , az are unit vectors in the ρ , ø , and z-directions as illustrated in Figure 1. The magnitude of $\vec{A}$ as:

$$|\vec{A}| = \sqrt{A_\rho^2 + A_\emptyset^2 + A_z^2}$$

A point *P* in cylindrical coordinates is represented as (ρ , ø, z) and is as shown in Figure 1. Observe Figure 1. closely and note how we define each space variable: ρ is the radius of the cylinder passing through P or the radial distance from the z-axis ø ; is (called the azimuthal angle) measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system

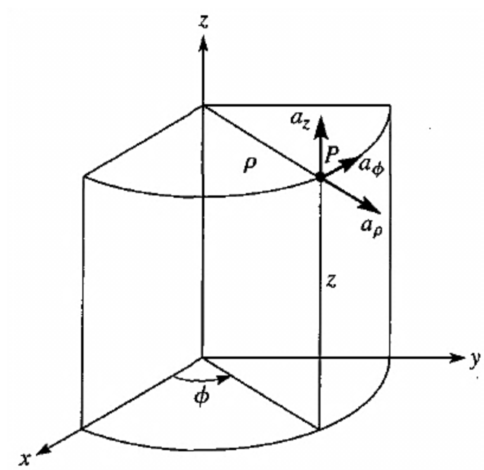


Figure 1. Point *P* and unit vectors in the cylindrical coordinate system

Notice that the unit vectors aρ, aø , az are mutually perpendicular since our coordinate system is orthogonal ; aρ points in the direction of increasing ρ, a ,ø , in the direction of increasing ø, and z in the positive z-direction. Thus

$$\boldsymbol{a}_\rho \cdot \boldsymbol{a}_\rho = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_\emptyset = \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1$$

$$\boldsymbol{a}_\rho \cdot \boldsymbol{a}_\emptyset = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_z = \boldsymbol{a}_z \cdot \boldsymbol{a}_\rho = 0$$

$$\boldsymbol{a}_\rho \times \boldsymbol{a}_\emptyset = \boldsymbol{a}_z$$

$$\boldsymbol{a}_\emptyset \times \boldsymbol{a}_z = \boldsymbol{a}_\rho$$

$$\boldsymbol{a}_z \times \boldsymbol{a}_\rho = \boldsymbol{a}_\emptyset$$

Note Also from Figure 1. that in cylindrical coordinate, differential element can be found:

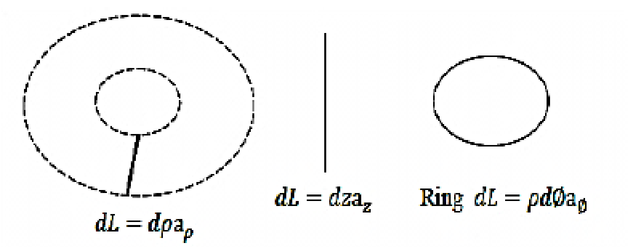
a. Differential displacement is given by:

$d\vec{L} = d\rho\ \boldsymbol{a}_\rho$

$d\vec{L} = \rho\ d\emptyset\ \boldsymbol{a}_\emptyset$

$d\vec{L} = dz\ \boldsymbol{a}_z$

or $d\vec{L} = d\rho\ \boldsymbol{a}_\rho + \rho\ d\emptyset\ \boldsymbol{a}_\emptyset + dz\ \boldsymbol{a}_z$

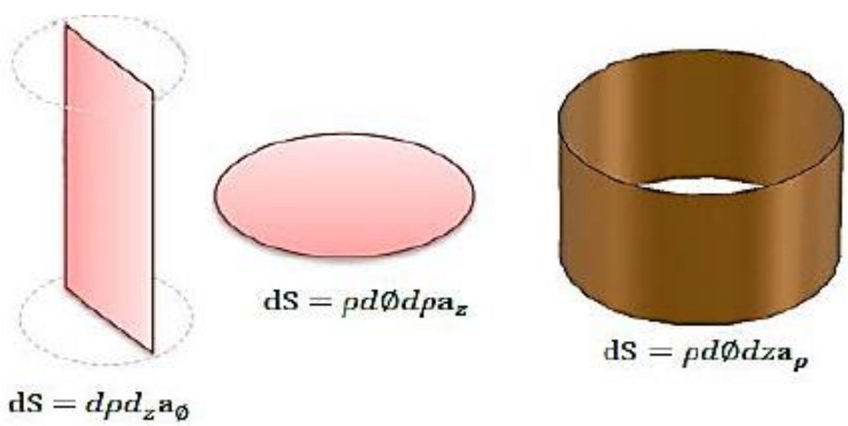


b. Differential normal area is given by:

$$d\vec{S} = \rho\ d\emptyset\ dz\ \boldsymbol{a}_\rho$$

$$d\vec{S} = d\rho\ dz\ \boldsymbol{a}_\emptyset$$

$$d\vec{S} = \rho\ d\rho\ d\emptyset\ \boldsymbol{a}_z$$



c. Differential volume is given by:

$$dV = \rho\ d\rho\ d\emptyset\ dz$$

d. The distance between two points in cylindrical coordinate P_1 (ρ_1 , $\emptyset_1$,z_1) and P_2(ρ_2, $\emptyset_2$,z_2)is given by

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 - 2\cos(\emptyset_2 - \emptyset_1) + (z_2 - z_1)^2}$$

- **<u>Cylindrical to Cartesian Coordinate Transformation</u>**

The relationships between the variables (ρ, ø, z) of the cylindrical coordinate system and those of the Cartesian system (x, y, z) are easily obtained as

$$x = \rho\ \cos\emptyset\,, \qquad y = \rho\ \sin\emptyset\,, \qquad z = z$$

In matrix form, we have transformation of vector $\vec{A}$

From cylindrical coordinate $\vec{A} = A_\rho \boldsymbol{a}_\rho + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_z$

To Cartesian coordinate $\vec{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z$ as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix}$$

- **<u>Cartesian to Cylindrical Coordinate Transformation</u>**

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ø, z) are easily obtained as

$$\rho = \sqrt{x^2 + y^2} \quad , \quad \emptyset = \tan^{-1}\frac{y}{x}, \quad z = z$$

In matrix form, we have transformation of vector $\vec{A}$

From Cartesian coordinate $\quad \vec{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z$

To cylindrical coordinate $\quad \vec{A} = A_\rho \boldsymbol{a}_\rho + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_z$ as

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Example: Given point $P(-2,6,3)$ and vector $\vec{A} = y\boldsymbol{a}_x + (x+z)\boldsymbol{a}_y$, express P and $\vec{A}$ in cylindrical coordinate. Evaluate $\vec{A}$ at P in Cartesian and cylindrical system?

Solution: *The vector* $\vec{A}$ *in Cartesian coordinate at P is:*

$$\vec{A} = 6\boldsymbol{a}_x + (-2+3)\boldsymbol{a}_y = 6\boldsymbol{a}_x + \boldsymbol{a}_y$$

$$|\vec{A}| = \sqrt{6^2 + 1^2} = 6.08$$

The point P in cylindrical coordinate is:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{2^2 + 6^2} = 6.324$$

$$\emptyset = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = z = 3$$

P(6.324 , 108.430°, 3)

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

But $x = \rho\ cos\,\emptyset\,,\ y = \rho\ sin\,\emptyset\,,\ z = z$ and substituting these yields

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\ \sin\emptyset \\ \rho\ \cos\emptyset + z \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \rho\ sin\,\emptyset\cos\emptyset + \rho\ \cos\emptyset\sin\emptyset + z\ \sin\emptyset \\ -\rho\sin^2\emptyset + \rho\cos^2\emptyset + z\,cos\,\emptyset \\ 0 \end{bmatrix}$$

$$A_\rho = \rho\ sin\,\emptyset\cos\emptyset + \rho\ \cos\emptyset\sin\emptyset + z\ \sin\emptyset$$

$$A_\emptyset = -\rho\sin^2\emptyset + \rho\cos^2\emptyset + z\,cos\,\emptyset$$

$$A_z = 0$$

$$\vec{A} = A_\rho \boldsymbol{a}_\rho + A_\emptyset \boldsymbol{a}_\emptyset$$

$\vec{A}$ *at point P is:*

$$A_\rho = 6.324\sin 108.43\cos 108.43 + 6.324\ \cos 108.43\sin 108.43 + 3\ \sin 108.43 = -0.948$$

$$A_\emptyset = -\,6.324\sin^2 108.43 + 6.324\ \cos^2 108.43 + 3\,cos\,108.43 = -6.008$$

$$\vec{A} = -0.948\boldsymbol{a}_\rho - 6.08\boldsymbol{a}_\emptyset$$

$$|\vec{A}| = \sqrt{(0.948)^2 + (6.08)^2} = 6.08$$

The Spherical Coordinate System (r, θ ,ø)

The Spherical coordinate system is most appropriate when dealing with problems having spherical symmetry. A vector $\vec{A}$in spherical coordinates can be written as:

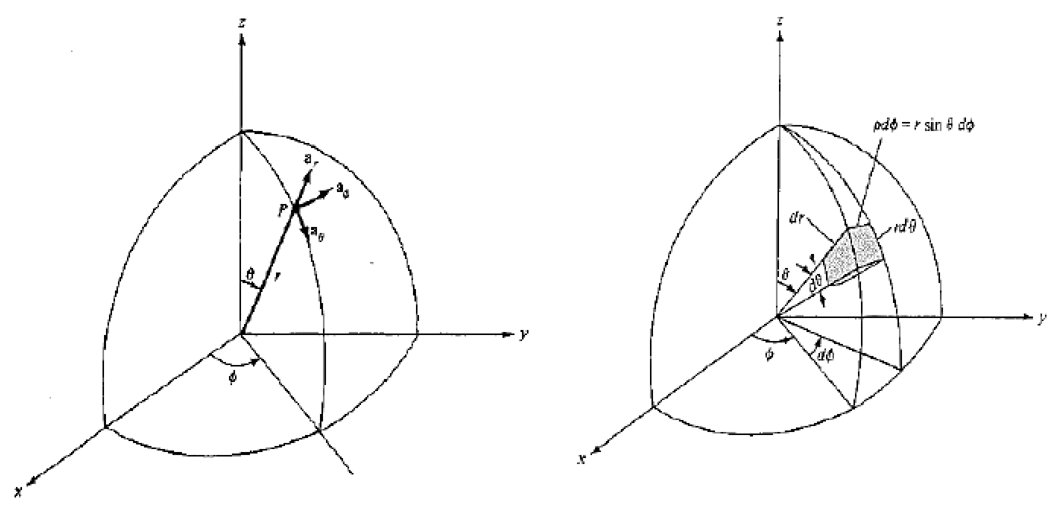
$$\vec{A} = A_r \boldsymbol{a}_r + +A_\theta \boldsymbol{a}_\theta + A_\emptyset \boldsymbol{a}_\emptyset$$

Where a_r, a_θ $a_\emptyset$, are unit vectors in the and -directions

The magnitude of $\vec{A}$ is:

$$|\vec{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\emptyset^2}$$

A point P in spherical coordinates is represented as (**r,θ,ø**) and is illustrate in Figure 1.6 (a). From this Figure, we notice that r is defined as the distance from the origin to the point P or the radius of a sphere centred at the origin and passing through P;θ is the angle between the z-axis and the position vector of P; and ø is measured from the x-axis



(a) (b)

Figure 1.6 Spherical coordinate system (a) Point P and unit vectors (b) Differential elements

Notice that the unit vectors a_r a_θ, and $a_\emptyset$ are mutually perpendicular since our coordinate system is orthogonal; a_r points in the direction of increasing r,a_θ in the direction of increasing θ, $a_\emptyset$ and in the direction of increasing ø . Thus

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_r = \boldsymbol{a}_\theta \cdot \boldsymbol{a}_\theta = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_\emptyset = 1$$

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_\theta = \boldsymbol{a}_\theta \cdot \boldsymbol{a}_\emptyset = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_r = 0$$

$$\boldsymbol{a}_r \times \boldsymbol{a}_\theta = \boldsymbol{a}_\emptyset$$

$$\boldsymbol{a}_\theta \times \boldsymbol{a}_\emptyset = \boldsymbol{a}_r$$

$$\boldsymbol{a}_\emptyset \times \boldsymbol{a}_r = \boldsymbol{a}_\theta$$

From Figure 1.6 (b), we note that in spherical coordinate, differential element can be found:

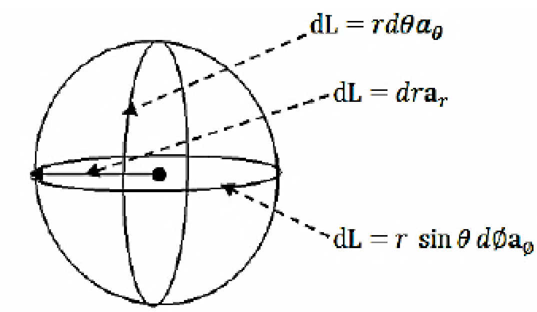
- Differential displacement is given by:

$$d\vec{L} = dr\, \boldsymbol{a}_r$$

$$d\vec{L} = r\, d\theta\, \boldsymbol{a}_\theta$$

$$d\vec{L} = r \sin\theta\, d\emptyset\, \boldsymbol{a}_\emptyset$$

Or $d\vec{L} = dr\, \boldsymbol{a}_r + r\, d\theta\, \boldsymbol{a}_\theta + r \sin\theta\, d\emptyset\, \boldsymbol{a}_\emptyset$

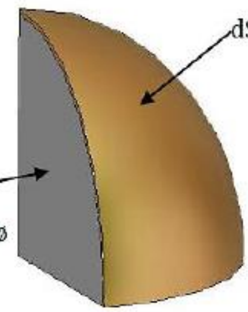


•Differential normal area is given by:

$d\vec{S} = r^2 \sin\theta \, d\theta \, d\emptyset \, \boldsymbol{a}_r$

$d\vec{S} = r \sin\theta \, dr \, d\emptyset \, \boldsymbol{a}_\theta$

$d\vec{S} = r \, dr \, d\theta \, \boldsymbol{a}_\emptyset$



•Differential volume is given by:

$dV = r^2 \sin\theta \, dr \, d\theta \, d\emptyset$

The distance between two points in spherical coordinate $P_1(r_1, \theta_1, \emptyset_1)$ and $P_2(r_2, \theta_2, \emptyset_2)$ is given

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\emptyset_2 - \emptyset_1)}$$

➢ **<u>Cartesian to Spherical Coordinate Transformation</u>**

The relationships between the variables (x, y , z) of the Cartesian coordinate system and those of the Spherical system (r, θ, ø) are easily obtained as

$$r = \sqrt{x^2 + y^2 + z^2} \qquad ,\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \emptyset = \tan^{-1}\frac{y}{x}$$

In matrix form, we have transformation of vector $\vec{A}$

From Cartesian coordinate $\vec{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z$

To Spherical coordinate $\vec{A} = A_r \boldsymbol{a}_r + A_\theta \boldsymbol{a}_\theta + A_\emptyset \boldsymbol{a}_\emptyset$ as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \sin\theta\sin\emptyset & \cos\theta \\ \cos\theta\cos\emptyset & \cos\theta\sin\emptyset & -\sin\theta \\ -\sin\emptyset & \cos\emptyset & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

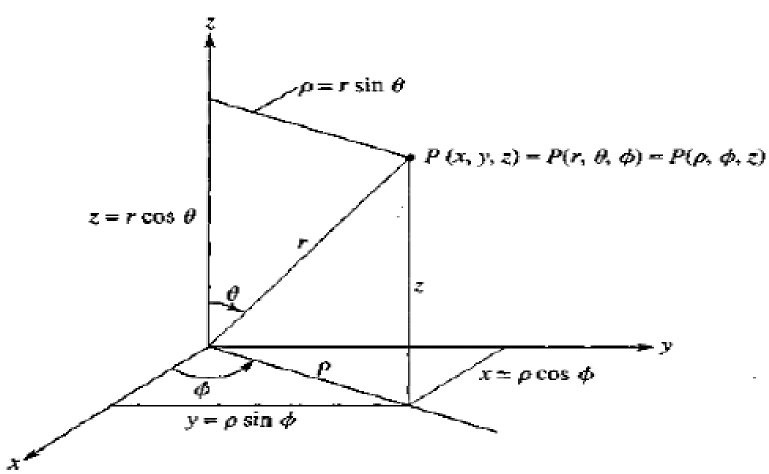


Figure 1.7: The relation between space variables (x, y, z) and (r, θ, ø)

- **<u>Spherical to Cartesian Coordinate Transformation</u>**

The relationships between the variables (r ,θ, ø) of the spherical coordinate system and those of the Cartesian system (x, y, z) are easily obtained as

$$x = r\sin\theta\cos\emptyset\,,\qquad y = r\sin\theta\sin\emptyset\,,\qquad z = r\cos\theta$$

In matrix form, we have transformation of vector $\vec{A}$

From Spherical coordinate $\quad \vec{A} = A_r\boldsymbol{a}_r + A_\theta\boldsymbol{a}_\theta + A_\emptyset\boldsymbol{a}_\emptyset$

To Cartesian coordinate $\quad \vec{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$ as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \cos\theta\cos\emptyset & -\sin\emptyset \\ \sin\theta\sin\emptyset & \cos\theta\sin\emptyset & \cos\emptyset \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix}$$

Example Given point P(—2, 6, 3) and vector A = ya_x + (x + z)a_y, express P and A in cylindrical and spherical coordinates. Evaluate A at P in the Cartesian, cylindrical, and spherical systems.

Solution: At point P: x =-2, y = 6, z = 3. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62°$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$$

In the Cartesian system, **A** at *P* is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

For vector **A**, $A_x = y, A_y = x + z, A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

or

$$A_\rho = y\cos\phi + (x + z)\sin\phi$$
$$A_\phi = -y\sin\phi + (x + z)\cos\phi$$
$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting these yields

$$\mathbf{A} = (A_\rho, A_\phi, A_z) = [\rho\cos\phi\sin\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi$$

At P

$$\rho = \sqrt{40}, \qquad \tan\phi = \frac{6}{-2}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}$$

$$\mathbf{A} = \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\rho + \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi$$
$$= \frac{-6}{\sqrt{40}}\mathbf{a}_\rho - \frac{38}{\sqrt{40}}\mathbf{a}_\phi = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi$$

Similarly, in the spherical system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

or

$$A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$

$$A_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$$

$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. Substituting these yields

$$\begin{aligned}\mathbf{A} &= (A_r, A_\theta, A_\phi) \\ &= r[\sin^2\theta\cos\phi\sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r \\ &\quad + r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta \\ &\quad + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi\end{aligned}$$

At P

$$r = 7, \qquad \tan\phi = \frac{6}{-2}, \qquad \tan\theta = \frac{\sqrt{40}}{3}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}, \qquad \cos\theta = \frac{3}{7}, \qquad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\mathbf{A} = 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_r$$

$$+ 7 \cdot \left[\frac{\sqrt{40}}{7} \cdot \frac{3}{7} \cdot \frac{6}{\sqrt{40}} \cdot \frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7} \right) \cdot \frac{3}{7} \cdot \frac{6}{\sqrt{40}} \right] \mathbf{a}_\theta$$

$$+ 7 \cdot \left[\frac{-\sqrt{40}}{7} \cdot \frac{36}{40} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7} \right) \cdot \frac{-2}{\sqrt{40}} \right] \mathbf{a}_\phi$$

$$= \frac{-6}{7} \mathbf{a}_r - \frac{18}{7\sqrt{40}} \mathbf{a}_\theta - \frac{38}{\sqrt{40}} \mathbf{a}_\phi$$

$$= -0.8571 \mathbf{a}_r - 0.4066 \mathbf{a}_\theta - 6.008 \mathbf{a}_\phi$$

Note that $|\mathbf{A}|$ is the same in the three systems; that is,

$$|\mathbf{A}(x, y, z)| = |\mathbf{A}(\rho, \phi, z)| = |\mathbf{A}(r, \theta, \phi)| = 6.083$$

3. Transfer the vector $\vec{A} = 10\, \boldsymbol{a}_x$ to spherical coordinate at point $P(x = -3, y = 2, z = 4)$

$$\vec{A} = -5.5702 \boldsymbol{a}_r - 6.18\, \boldsymbol{a}_\theta - 5.547 \boldsymbol{a}_\emptyset$$

4. Give the Cartesian coordinates of $\vec{H} = 20\, \boldsymbol{a}_\rho - 10\, \boldsymbol{a}_\emptyset + 3\, \boldsymbol{a}_z$ at point $P(x = 5, y = 2, z = -1)$

$$\vec{H} = 22.282\, \boldsymbol{a}_x - 1.856\, \boldsymbol{a}_y + 3\, \boldsymbol{a}_z$$