Electric Fields Duo to Continuous Charge Distribution

So far we have only considered forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in Figure .1

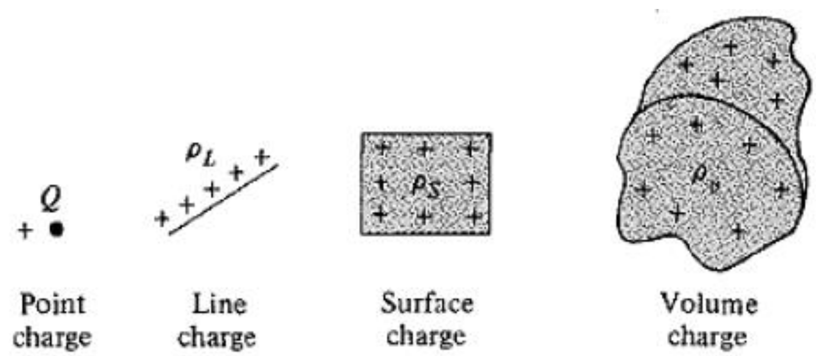


Figure 1 Various charge distributions and charge elements.

It is customary to denote the

- ***line charge*** density by $\boldsymbol{\rho_L}$ (in $\mathbf{C/m}$),
- ***surface charge*** density by $\boldsymbol{\rho_S}$ (in $\mathbf{C/m^2}$), and
- ***volume charge*** density by $\boldsymbol{\rho_v}$ (in $\mathbf{C/m^3}$)

The ***electric field intensity*** due to each of the charge distributions ρ_L , ρ_s , and ρ_v are given by

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R \qquad (\textit{line charge})$$

$$\vec{E} = \int_S \frac{\rho_S \, ds}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R \qquad (\textit{surface charge})$$

$$\vec{E} = \int_v \frac{\rho_v \, dv}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R \qquad (\textit{volume charge})$$

a. Field of a Line Charge

Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in Figure 3.4 The charge element d_Q associated with element dl=dz of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz$$

from Figure 3.4

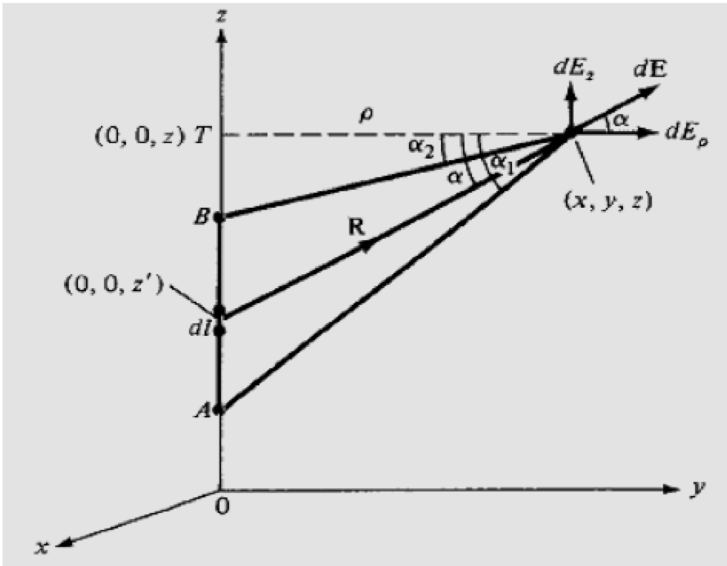


Figure 3.4 Evaluation of the E field due to a line charge

$$dl = dz'$$

$$\vec{R} = (x, y, z) - (0, 0, z')$$

$$= x\boldsymbol{a}_x + y\boldsymbol{a}_y + (z - z')\boldsymbol{a}_z$$

or

$$\vec{R} = \rho\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z$$

$$R^2 = \left|\vec{R}\right|^2$$

$$= x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$

$$R = \sqrt{\rho^2 + (z - z')^2}$$

$$\frac{\mathbf{a}_R}{R^2} = \frac{\vec{R}}{\left|\vec{R}\right|^3} = \frac{\rho\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z}{\sqrt{\rho^2 + (z - z')^2}\ [\rho^2 + (z - z')^2]^2} = \frac{\rho\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

Substituting all this into eq. of ***electric field intensity*** we get

$$\vec{\mathrm{E}} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R$$

$$\vec{\mathrm{E}} = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho \boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} \, dz' \qquad (*)$$

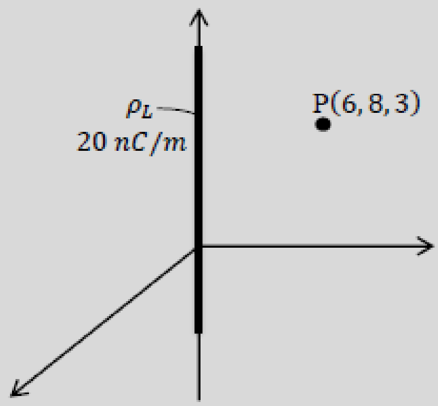
As a special case, for an infinite line charge, point B is at(0,0,∞) and A at (0,0,-∞) so that the equation. Becomes

$$\vec{\mathrm{E}} = \frac{\rho_L}{2\pi\varepsilon_0\rho} \mathbf{a}_\rho$$

Example: A uniform line charge, infinite in extent with $\rho_L = 20\ nC/m$ lies along $z-axis$. Find the $\vec{E}$ at $(6,8,3)m$.

Solution:

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$



$$\vec{\rho} = (6,8,3) - (0,0,3)$$

$$\vec{\rho} = 6\ \mathbf{a}_x + 8\ \mathbf{a}_y$$

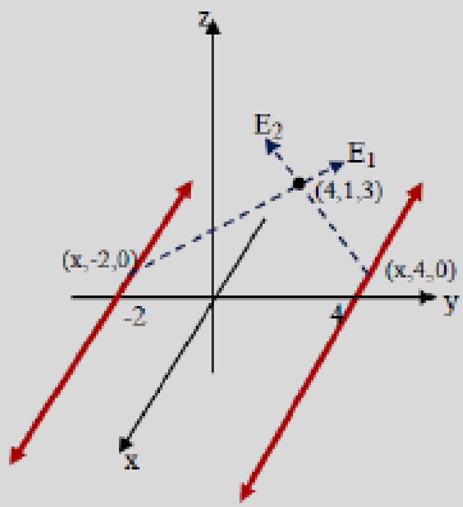
$$|\vec{\rho}| = \sqrt{6^2 + 8^2} = 10$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{6\ \mathbf{a}_x + 8\ \mathbf{a}_y}{10} = 0.6\ \mathbf{a}_x + 0.8\ \mathbf{a}_y$$

$$\vec{E} = \frac{20\times10^{-9}}{2\pi\times8.854\times10^{-12}\times10}[0.6\ \mathbf{a}_x + 0.8\ \mathbf{a}_y] = 21.571\ \mathbf{a}_x + 28.761\ \mathbf{a}_y\ \text{V/m}$$

Example: Two uniform line charges of $\rho_L = 5\ nC/m$ each are parallel to the $x\ axis$, one at $z = 0, y = -2\ m$ and the other at $z = 0, y = 4m$. Find $\vec{E}$ at $(4, 1, 3)\ m$?

Solution:



$\vec{E} = \vec{E}_1 + \vec{E}_2$

$$\vec{E}_1 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

$\vec{\rho}_1 = (1 - (-2))\ \mathbf{a}_y + (3\ - 0)\ \mathbf{a}_z$

$\vec{\rho}_1\ = 3\ \mathbf{a}_y + 3\ \mathbf{a}_z$

$|\vec{\rho}_1| = \sqrt{3^2 + 3^2} = \sqrt{18}$

$$\mathbf{a}_\rho = \frac{\vec{\rho}_1}{|\vec{\rho}_1|} = \frac{3\ \mathbf{a}_y + 3\ \mathbf{a}_z}{\sqrt{18}}$$

$$\vec{E}_1 = \frac{5 \times 10^{-9}}{2\pi \times 8.8541 \times 10^{-12}}\left[\frac{3\ \mathbf{a}_y + 3\ \mathbf{a}_z}{18}\right]\text{V/m}$$

$$\vec{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\,\mathbf{a}_\rho$$

$$\vec{\rho} = (1-4)\,\mathbf{a}_y + (3\ -0)\,\mathbf{a}_z$$

$$\vec{\rho}\ \ = -3\,\mathbf{a}_y + 3\,\mathbf{a}_z$$

$$|\vec{\rho}| = \sqrt{3^2+3^2} = \sqrt{18}$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{-3\,\mathbf{a}_y + 3\,\mathbf{a}_z}{\sqrt{18}}$$

$$\vec{E}_2 = \frac{5\times10^{-9}}{2\pi\times8.8541\times0^{-12}}\left[\frac{-3\,\mathbf{a}_y + 3\,\mathbf{a}_z}{18}\right]\ \mathrm{V/m}$$

$$\vec{E} = 2 * \frac{5\times10^{-9}}{2\pi\times8.8541\times10^{-12}}\,\frac{6\,\mathbf{a}_z}{18} = 30\,\mathbf{a}_z\ \mathrm{V/m}$$

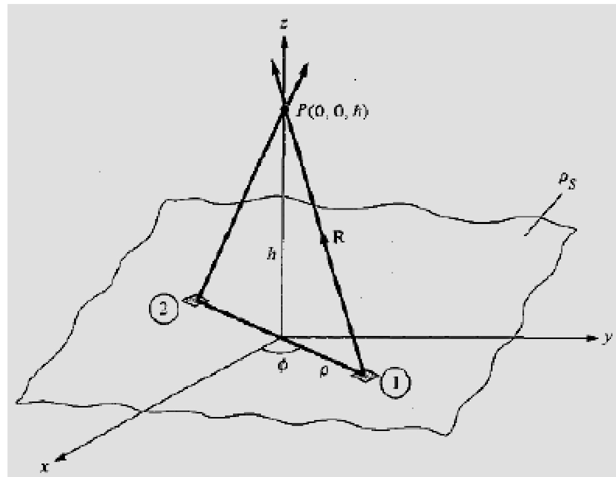
b. Field of a Sheet Charge

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_S. The charge associated with an elemental area dS is

$$dQ = \rho_S \, dS$$

from the eq.

$$\vec{E} = \int_S \frac{\rho_S \, dS}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R$$



From eq. above, the contribution to the ¯E field at point P(0, 0, h) by the elemental surface 1 shown in Figure

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R (***)$$

…………………………………..(1)

From figure

$$\vec{R} = \rho\,(-\mathbf{a}_\rho) + h\,\mathbf{a}_z$$

$$= -\mathbf{a}_\rho \rho + h\,\mathbf{a}_z$$

$$|\vec{R}| = [\rho^2 + h^2]^{1/2}$$

$$\mathbf{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-\mathbf{a}_\rho \rho + h\,\mathbf{a}_z}{[\rho^2 + h^2]^{1/2}}$$

$$dQ = \rho_S\, dS = \rho_S\, \rho\, d\emptyset\, d\rho$$

substitution of these terms into eq. (***) gives

$$d\vec{E} = \frac{\rho_S\, \rho\, d\emptyset\, d\rho}{4\pi\varepsilon_0[\rho^2 + h^2]}\left[\frac{-\mathbf{a}_\rho \rho + h\,\mathbf{a}_z}{[\rho^2 + h^2]^{1/2}}\right]$$

$$= \frac{\rho_S\, \rho\, d\emptyset\, d\rho\,[-\mathbf{a}_\rho \rho + h\,\mathbf{a}_z]}{4\pi\varepsilon_0[\rho^2 + h^2]^{3/2}}$$

$$d\vec{E} = d\vec{E}_\rho + d\vec{E}_z$$

Since $d\vec{E}_\rho = 0$ from the symmetry of the charge distribution,

$$d\vec{E} = \frac{\rho_S\, h\, \rho\, d\emptyset\, d\rho}{4\pi\varepsilon_0[\rho^2 + h^2]^{3/2}}\, \mathbf{a}_z$$

$$\vec{E} = \int_S d\vec{E}_z = \frac{\rho_S}{4\pi\varepsilon_0}\int_{\emptyset=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\,\rho\, d\emptyset\, d\rho}{[\rho^2 + h^2]^{3/2}}\, \mathbf{a}_z$$

$$\vec{E} = \frac{\rho_S\, h}{4\pi\varepsilon_0}\, 2\pi\int_{\rho=0}^{\infty}[\rho^2 + h^2]^{-3/2}\,\frac{1}{2}\, d(\rho^2)\, \mathbf{a}_z$$

$$= \frac{\rho_S\, h}{2\varepsilon_0}\left\{-[\rho^2 + h^2]^{-1/2}\right\}_0^{\infty} \mathbf{a}_z$$

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\, \mathbf{a}_z$$

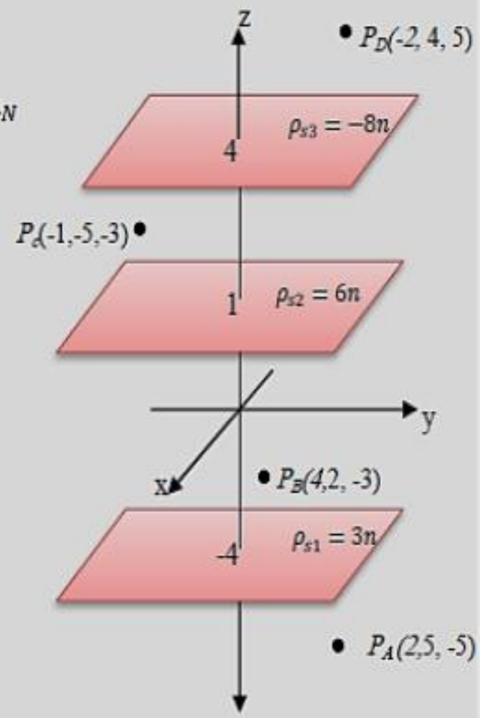
for an *infinite sheet* of charge

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\, \mathbf{a}_N$$

Example: Three infinite uniform sheets of charge are located in free space as follows: $3nC/m^2$ at $z = -4$, $6nC/m^2$ at $z = 1$, and $-8nC/m^2$ at $z = 4$. Find E at the point: (a) $P_A(2, 5, -5)$; (b) $P_B(4, 2, -3)$; (c) $P_c(-1, -5, 2)$; (d) $P_D(-2, 4, 5)$?

Solution:

$a -$ at p_A



$Becuse\ the\ infinite\ sheet\ charge\ the\ E = \vec{E} = \frac{\rho_S}{2\varepsilon_o} a_N$

$$E_T = \left[\frac{-3n}{2\varepsilon_0} - \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right] a_z = -56.5\, a_z\ V/m$$

b- at p_B

$$E = \left[\frac{3n}{2\varepsilon_0} - \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right] a_z = 282.3\, a_z\ V/m$$

c- at p_C

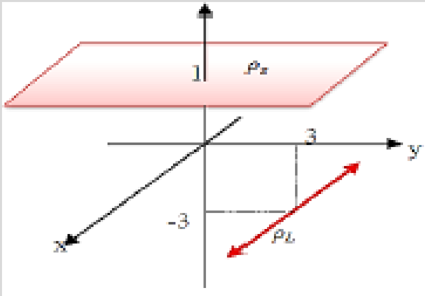
$$E = \left[\frac{3n}{2\varepsilon_0} + \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right] a_z = 960.45 a_z\ V/m$$

$d -$ at p_d

$$E = \left[\frac{3n}{2\varepsilon_0} + \frac{6n}{2\varepsilon_0} - \frac{8n}{2\varepsilon_0}\right] a_z = 56.5 a_z\ V/m$$

Example: A uniform sheet charge with $\rho_S = 1/3\pi\ nC/m^2$ is located at $z = 5\ m$ and a uniform line charge with $\rho_L = 25/9\ nC/m$ at $y = 3m$ and $z = -3\ m$. Find $\vec{E}$ at $(x, -1, 0)m$

Solution:



$$\vec{E}_T = \vec{E}_1 + \vec{E}_2$$

$\vec{E}_1$ *due to surface charge*

$\vec{E}_2$ *due to line charge*

$$\vec{E}_1 = \frac{\rho_S}{2\varepsilon_0}\,\mathbf{a}_N$$

$$\vec{E}_1 = \frac{(1/3\pi)\times 10^{-9}}{2\varepsilon_0}(-\,\boldsymbol{a}_z) = -6\,\boldsymbol{a}_z$$

$$\vec{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\,\mathbf{a}_\rho$$

$$\vec{\rho} = (x, -1, 0) - (x, 3, -3) = (-1-3)\,\mathbf{a}_y + (0\ -(-3))\,\mathbf{a}_z$$

$$|\vec{\rho}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{-4\,\boldsymbol{a}_y + 3\,\boldsymbol{a}_z}{5}$$

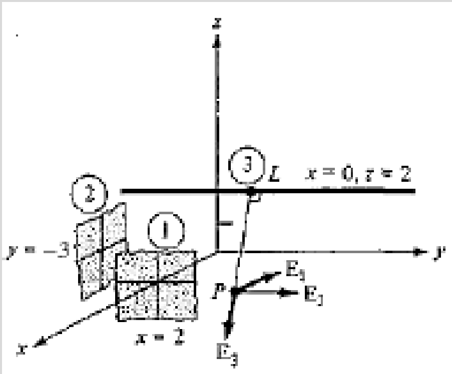
$$\vec{E}_2 = \frac{(25/9)\times 10^{-9}}{2\pi\times 8.854\times 10^{-12}\times 5}\left[\frac{-4\,\boldsymbol{a}_y + 3\,\boldsymbol{a}_z}{5}\right]$$

$$\vec{E}_2 = 8\,\boldsymbol{a}_y + 6\,\boldsymbol{a}_z$$

$$\vec{E}_T = -6\,\boldsymbol{a}_z + 8\,\boldsymbol{a}_y + 6\,\boldsymbol{a}_z = 8\,\boldsymbol{a}_y \qquad V/m$$

Example: Planes $x = 2$ and $y = -3$, respectively, carry charges $10\ nC/m^2$ and $15\ nC/m^2$. If the line $x = 0, z = 2$ carries charge $10\pi\ nC/m$, calculate $\boldsymbol{E}$ at $(1, 1, -1)$ due to the three charge distributions.

Solution:



$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\ \mathbf{a}_N$$

$$\vec{E}_1 = \frac{\rho_{S_1}}{2\varepsilon_0}\ (-\boldsymbol{a}_x) = -\frac{10\cdot 10^{-9}}{2\cdot\frac{10^{-9}}{36\pi}}\ \boldsymbol{a}_x = -180\pi\ \ \boldsymbol{a}_x$$

$$\vec{E}_2 = \frac{\rho_{S_2}}{2\varepsilon_0}\ \boldsymbol{a}_y = \frac{15\cdot 10^{-9}}{2\cdot\frac{10^{-9}}{36\pi}}\ \boldsymbol{a}_y = 270\pi\ \ \boldsymbol{a}_y$$

$$\vec{E}_3 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

$$\vec{\rho}\ = (1,1,-1) - (0,1,2) =\ \mathbf{a}_x - 3\ \mathbf{a}_z$$

$$|\vec{\rho}| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{\boldsymbol{a}_x - 3\,\boldsymbol{a}_z}{\sqrt{10}}$$

$$\vec{E}_3 = \frac{10\pi \cdot 10^{-9}}{2\pi \cdot \frac{10^{-9}}{36\pi}\,\sqrt{10}} \cdot \frac{\boldsymbol{a}_x - 3\,\boldsymbol{a}_z}{\sqrt{10}} = 18\pi(\,\boldsymbol{a}_x - 3\,\boldsymbol{a}_z)$$

$$\vec{E} = -180\pi\ \boldsymbol{a}_x + 270\pi\ \boldsymbol{a}_y + 18\pi(\,\boldsymbol{a}_x - 3\,\boldsymbol{a}_z)$$

$$= -162\pi\ \boldsymbol{a}_x + 270\pi\ \boldsymbol{a}_y - 54\pi\,\boldsymbol{a}_z\ \ V/m$$

C. Field Due to a Continuous Volume Charge Distribution

If we now visualize a region of space filled with a great number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a volume charge density ρ_v C/m^3

اذا تصورنا منطقة من الفراغ مملوءة بعدد هائل من الشحنات المنفصلة عن بعضها بمسافات صغيرة جدا فاننا نستطيع إحلال هذا التوزيع لجسميات صغيرة بتوزيع املس يوصف بكثافة شحنة حجمية

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{vol} \rho_v \, dv$$

$$\vec{E} = \int_{vol} \frac{\rho_v \, dv}{4\pi\varepsilon_0 |\vec{R}|^2} \boldsymbol{a}_R$$

Example: Calculate the total charge within each of the indicate volumes:

(a) $0.1 \le x, y, z \le 0.2$ $\quad \rho_v = \frac{1}{x^3 y^3}$

(b) $0 \le \rho \le 0.1 \quad 0 \le \emptyset \le \pi \quad 2 \le z \le 4$ $\quad \rho_v = \rho^2 z^2 \sin 0.6\emptyset$

(c) $Univers$ $\quad \rho_v = \frac{e^{-2r}}{r^2}$

Solution:

(a)

$$Q = \int_{vol} \rho_v \, dv \qquad dv = dx\, dy\, dz$$

$$Q = \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \int_{x=0.1}^{0.2} \frac{1}{x^3 y^3}\, dx\, dy\, dz = \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \int_{x=0.1}^{0.2} x^{-3} y^{-3}\, dx\, dy\, dz$$

$$Q = \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \left[-\frac{1}{2x^2}\right]_{0.1}^{0.2} \frac{1}{y^3}\, dy\, dz$$

$$= \left[-\frac{1}{2x^2}\right]_{0.1}^{0.2} \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \frac{1}{y^3}\, dy\, dz$$

$$=\left[-\frac{1}{2x^2}\right]_{0.1}^{0.2}\left[-\frac{1}{2y^2}\right]_{0.1}^{0.2}\int_{z=0.1}^{0.2} dz=\left[-\frac{1}{2x^2}\right]_{0.1}^{0.2}\left[-\frac{1}{2y^2}\right]_{0.1}^{0.2}[z]_{0.1}^{0.2}$$

$$=\left[-\frac{1}{2(0.2)^2}+\frac{1}{2(0.1)^2}\right]\left[-\frac{1}{2(0.2)^2}+\frac{1}{2(0.1)^2}\right][0.2-0.1]$$

$Q = 140.6\ C$

(b)

$$Q = \int_{vol} \rho_v \, dv \qquad dv = \rho \, d\rho \, d\emptyset \, dz$$

$$Q = \int_{z=2}^{4} \int_{\emptyset=0}^{\pi} \int_{\rho=0}^{0.1} \rho^2 z^2 \sin 0.6\emptyset \;\; \rho \, d\rho \, d\emptyset \, dz$$

$$= \int_{z=2}^{4} \int_{\emptyset=0}^{\pi} \int_{\rho=0}^{0.1} \rho^3 z^2 \sin 0.6\emptyset \;\; d\rho \, d\emptyset \, dz$$

$$= \int_{z=2}^{4} \int_{\emptyset=0}^{\pi} \left[\frac{\rho^4}{4}\right]_0^{0.1} z^2 \sin 0.6\emptyset \;\; d\emptyset \, dz = \int_{z=2}^{4} \left[\frac{\rho^4}{4}\right]_0^{0.1} \left[\frac{-\cos 0.6\emptyset}{0.6}\right]_0^{\pi} z^2 \, dz$$

$$= \left[\frac{\rho^4}{4}\right]_0^{0.1} \left[\frac{-\cos 0.6\emptyset}{0.6}\right]_0^{\pi} \left[\frac{z^3}{3}\right]_2^{4} = 1.018 \;\; mC$$

H.W:

On the line $x = 4$ and $y = -4$, there is a uniform charge distribution with density $\rho_L = 25 \ \frac{nC}{m}$. Determine $\vec{E}$ at $(-2, -1, 4)m$.

Ans: $\vec{E} = -59.92 \, \boldsymbol{a_x} + 29.969 \;\; \boldsymbol{a_y} \;\; V/m$