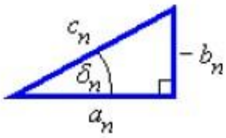
Fourier Transforms

Frequency Spectrum

The Fourier series may be combined into a single cosine series. Let p be the fundamental period. If the function f (x) is not periodic at all on [–L, L], then the fundamental period of the extension of f (x) to the entire real line p=2L

Let the **phase** angle δ_n be such that $\tan\delta_n = -\dfrac{b_n}{a_n}$,

so that $\sin\delta_n = -\dfrac{b_n}{c_n}$ and $\cos\delta_n = +\dfrac{a_n}{c_n}$



where the **amplitude** is $c_n = \sqrt{a_n^2 + b_n^2}$.

Also, in the trigonometric identity $\cos A\cos B - \sin A\sin B \equiv \cos(A+B)$,

replace A by $n\omega x$ and B by δ_n. Then

$$a_n\cos(n\omega x) + b_n\sin(n\omega x) = (c_n\cos\delta_n)\cos(n\omega x) - (c_n\sin\delta_n)\sin(n\omega x)$$

$$= c_n\cos(n\omega x + \delta_n), \quad \text{where} \quad \boxed{\omega = \frac{2\pi}{p} = \frac{\pi}{L}}, \quad \boxed{c_n = \sqrt{a_n^2 + b_n^2}} \quad \text{and} \quad \boxed{\tan\delta_n = -\frac{b_n}{a_n}}$$

Therefore the phase angle or **harmonic** form of the Fourier series is

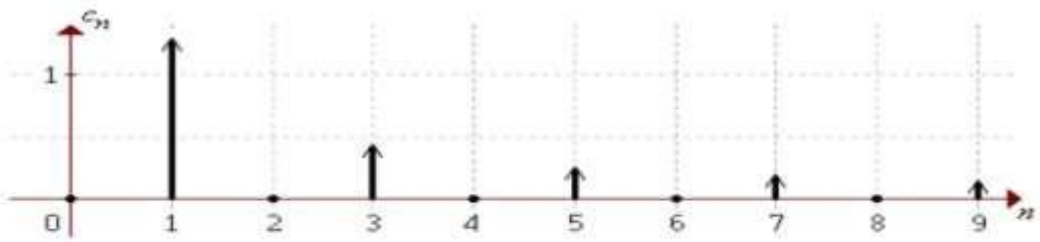
$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n\cos(n\omega x + \delta_n)}$$

Example: Plot the frequency spectrum for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

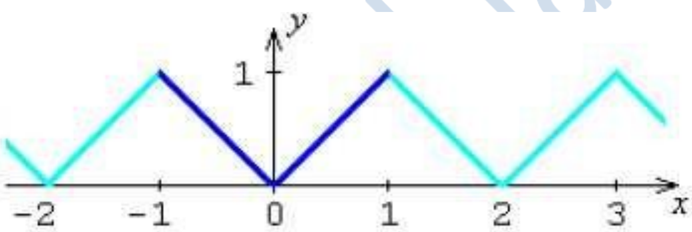
The Fourier series for the standard square wave is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

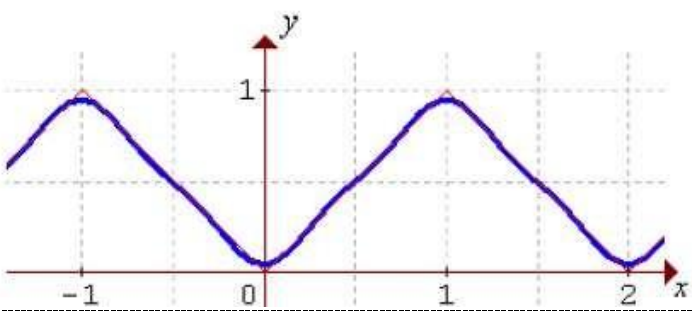


Example: Plot the frequency spectrum for the periodic extension of

$$f(x) = |x|, \quad -1 < x < 1$$



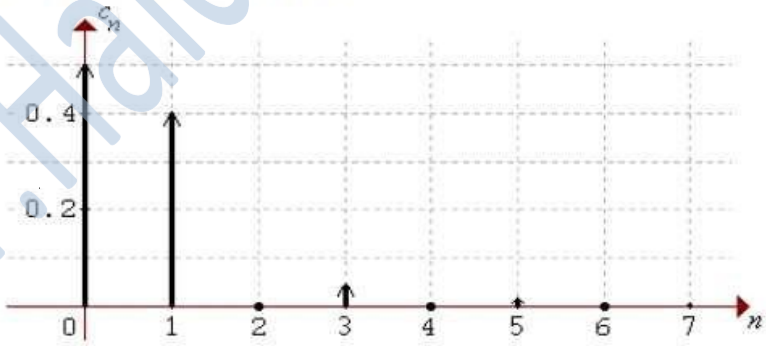
(which converges very rapidly, as this third partial sum demonstrates)



The harmonic amplitudes are

$$c_n = \begin{cases} \frac{1}{2} & (n=0) \\ \frac{2\left(1-(-1)^n\right)}{(n\pi)^2} & (n \in \mathbb{N}) \end{cases} = \begin{cases} \frac{1}{2} & (n=0) \\ 0 & (n \text{ even}, n \ge 2) \\ \frac{4}{(n\pi)^2} & (n \text{ odd}) \end{cases}$$

The frequencies therefore diminish rapidly:



Fourier Integrals

The Fourier series may be extended from $(-L, L)$ to the entire real line.

Let $\omega_n = \frac{n\pi}{L} \quad \Rightarrow \quad \omega_n - \omega_{n-1} = \frac{\pi}{L} = \Delta\omega \quad \Rightarrow \quad \frac{1}{L} = \frac{\Delta\omega}{\pi}$

The Fourier series for $f(x)$ on $(-L, L)$ is

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + $$

$$\sum_{n=1}^{\infty}\left(\frac{1}{L}\left(\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\right)\cos\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. + \frac{1}{L}\left(\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt\right)\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\Rightarrow \quad f(x) = \frac{\Delta\omega}{2\pi}\int_{-L}^{L} f(t)\,dt + $$

$$\sum_{n=1}^{\infty}\left(\frac{\Delta\omega}{\pi}\left(\int_{-L}^{L} f(t)\cos(\omega_n t)\,dt\right)\cos(\omega_n x)\right.$$

$$\left. + \frac{\Delta\omega}{\pi}\left(\int_{-L}^{L} f(t)\sin(\omega_n t)\,dt\right)\sin(\omega_n x)\right)$$

Email: haider.jabber@uomus.edu.iq

Now take the limit as $L \to \infty \quad \Rightarrow \quad \Delta\omega \to 0$:

The first integral converges to some finite number, so the first term vanishes in the limit.

The summation becomes an integral over all frequencies in the limit:

$$f(x) \to 0 +$$

$$\int_0^{\infty} \left(\frac{1}{\pi} \left(\int_{-\infty}^{\infty} f(t) \cos(\omega t)\, dt \right) \cos(\omega x)\, d\omega \right.$$

$$\left. + \frac{1}{\pi} \left(\int_{-\infty}^{\infty} f(t) \sin(\omega t)\, dt \right) \sin(\omega x)\, d\omega \right)$$

Therefore the Fourier integral of $f(x)$ is

$$\boxed{f(x) = \int_0^{\infty} \left(A_\omega \cos(\omega x) + B_\omega \sin(\omega x) \right) d\omega}$$

where the Fourier integral coefficients are

$$\boxed{A_\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega t)\, dt \quad \text{and} \quad B_\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega t)\, dt}$$

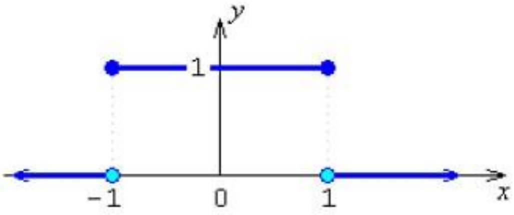
provided $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

Example: Find the Fourier integral of

$$f(x) = \begin{cases} 1 & (-1 \le x \le +1) \\ 0 & (\text{otherwise}) \end{cases}$$

Email: haider.jabber@uomus.edu.iq

From the functional form and from the graph of $f(x)$, it is obvious that $f(x)$ is piecewise smooth and that $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges to the value



$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt = \frac{1}{\pi}\int_{-1}^{1}\cos(\omega t)\,dt = \frac{1}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

The function $f(x)$ is even $\Rightarrow B_\omega = 0$ for all ω.

Therefore the Fourier integral of $f(x)$ is

$$\boxed{f(x) = \int_0^{\infty}\frac{2\sin\omega}{\pi\omega}\cos(\omega x)\,d\omega}$$

It also follows that

$$\int_0^{\infty}\frac{2\sin\omega}{\pi\omega}\cos(\omega x)\,d\omega = \begin{cases} 1 & (-1 < x < 1) \\ \frac{1}{2} & (x = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Fourier series and Fourier integrals can be used to evaluate summations and definite integrals that would otherwise be difficult or impossible to evaluate. For example, setting $x = 0$ in Example 7.06.1, we find that

$$\int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Complex Fourier Integrals

$$f(x) = \int_0^{\infty}\left(A_\omega\cos(\omega x) + B_\omega\sin(\omega x)\right)d\omega$$

$$= \int_0^{\infty}\left(A_\omega\frac{e^{j\omega x}+e^{-j\omega x}}{2} + B_\omega\frac{e^{j\omega x}-e^{-j\omega x}}{2j}\right)d\omega$$

$$= \int_0^{\infty}\left(\left(\frac{A_\omega - jB_\omega}{2}\right)e^{j\omega x} + \left(\frac{A_\omega + jB_\omega}{2}\right)e^{-j\omega x}\right)d\omega$$

$$= \int_0^{\infty}\left(C_\omega e^{j\omega x} + C_\omega^{*} e^{-j\omega x}\right)d\omega, \quad \text{where} \quad C_\omega = \frac{A_\omega - jB_\omega}{2}$$

But $C_{\omega}^{*} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\cos(\omega t) + j\sin(\omega t)}{2}\,dt$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\cos(-\omega t) - j\sin(-\omega t)}{2}\,dt = C_{-\omega}$$

and $\int_{0}^{\infty}\left(C_{-\omega}e^{-j\omega x}\right)d\omega = \int_{-\infty}^{0}\left(C_{+\omega}e^{+j\omega x}\right)d\omega$

By convention, the factor of $\frac{1}{2\pi}$ is extracted from the coefficients.

Therefore the complex Fourier integral of $f(t)$ is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{\omega}\,e^{j\omega t}\,dt$$

where the complex Fourier integral coefficients are

$$C_{\omega} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

(which is also the **Fourier transform** of f. $f(\omega) = \mathcal{F}[f(t)](\omega)$).

ω is the **frequency** of the signal $f(t)$.

Email: haider.jabber@uomus.edu.iq

Fourier Transforms

If $\hat{f}(\omega)$ is the Fourier transform of $f(t)$, then

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

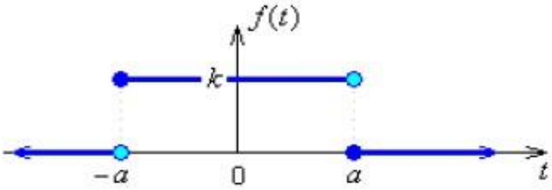
and the inverse Fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega t}\, d\omega$$

Example: Find the Fourier transform of the pulse function

$$f(t) = k\left(H(t+a) - H(t-a)\right) = \begin{cases} k & (-a \le t < a) \\ 0 & (\text{otherwise}) \end{cases}$$

From the functional form and from the graph of $f(t)$, it is obvious that $f(t)$ is piecewise smooth and that $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges to the value $2ak$.



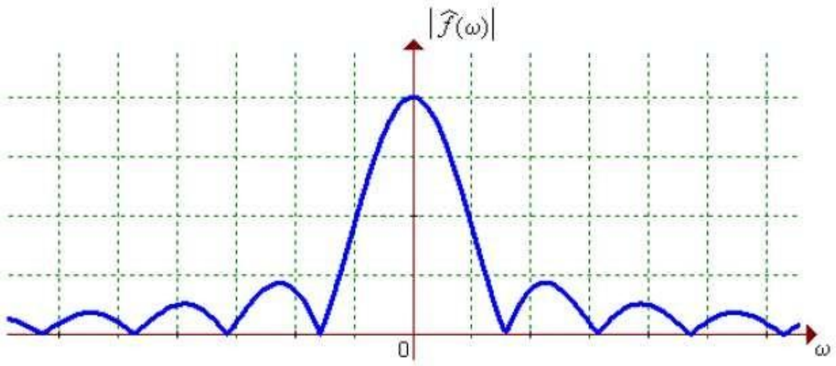
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{-a}^{a} k e^{-j\omega t}\, dt = \left[\frac{k e^{-j\omega t}}{-j\omega}\right]_{-a}^{a} = \frac{2k}{\omega} \cdot \frac{\left(-e^{-j\omega a} + e^{+j\omega a}\right)}{2j}$$

Therefore

$$\hat{f}(\omega) = 2k \frac{\sin(a\omega)}{\omega}$$

Email: haider.jabber@uomus.edu.iq

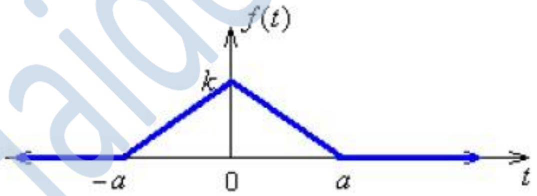
The transform is real. Therefore the frequency spectrum follows quickly:



Example: Find the Fourier transform of the triangle function

$$f(t) = \begin{cases} \frac{k}{a}(a-|t|) & (-a \le t \le a) \\ 0 & (\text{otherwise}) \end{cases}$$

From the functional form and from the graph of $f(t)$, it is obvious that $f(t)$ is piecewise smooth and that $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges to the value ak.



Email: haider.jabber@uomus.edu.iq

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \frac{k}{a}\int_{-a}^{0}(a+t)e^{-j\omega t}\,dt + \frac{k}{a}\int_{0}^{a}(a-t)e^{-j\omega t}\,dt$$

$$= \frac{k}{a}\left[\left(\frac{a+t}{-j\omega} - \frac{1}{(-j\omega)^2}\right)e^{-j\omega t}\right]_{-a}^{0} + \frac{k}{a}\left[\left(\frac{a-t}{-j\omega} - \frac{-1}{(-j\omega)^2}\right)e^{-j\omega t}\right]_{0}^{a}$$

$$= \frac{k}{a}\left[\left(\frac{ja}{\omega} + \frac{1}{\omega^2}\right) - \left(0 + \frac{1}{\omega^2}\right)e^{j\omega a} + \left(0 - \frac{1}{\omega^2}\right)e^{-j\omega a} - \left(\frac{ja}{\omega} - \frac{1}{\omega^2}\right)\right]$$

$$= \frac{k}{a\omega^2}\left(2 - e^{j\omega a} - e^{-j\omega a}\right) = \frac{2k}{a\omega^2}\left(1 - \cos(a\omega)\right)$$

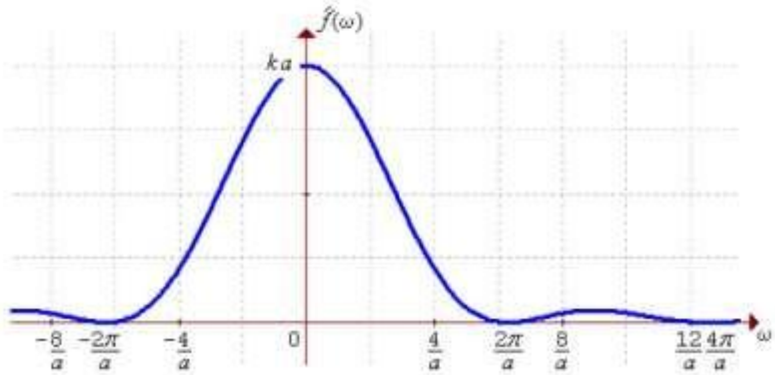
Therefore

$$\boxed{\hat{f}(\omega) = \frac{2k}{a}\cdot\frac{1-\cos(a\omega)}{\omega^2}}$$

An equivalent form is

$$\boxed{\hat{f}(\omega) = ak\cdot\left(\frac{\sin\left(\frac{a\omega}{2}\right)}{\left(\frac{a\omega}{2}\right)}\right)^2}$$

The Fourier transform of the triangle function happens to be real and non-negative, so that it is its own frequency spectrum $\left|\hat{f}(\omega)\right| = \hat{f}(\omega)$.



Email: haider.jabber@uomus.edu.iq