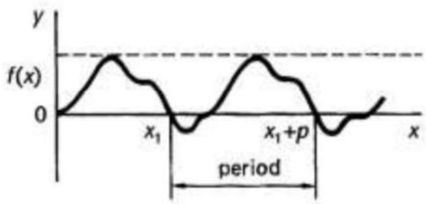
Fourier Series

We have seen earlier that many functions can be expressed in the form of infinite series. Problems involving various forms of oscillations are common in fields of modern technology and *Fourier series*, with which we shall now be concerned, enable us to represent a periodic function as an infinite trigonometrical series in sine and cosine terms. One important advantage of a Fourier series is that it can represent a function containing discontinuities, whereas Maclaurin's and Taylor's series require the function to be continuous throughout.

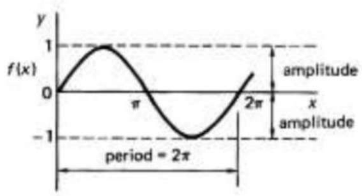
Periodic functions

A function $f(x)$ is said to be *periodic* if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the *period* of the oscillations.

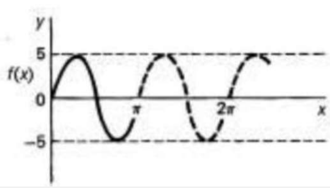


Graphs of $y = A \sin nx$

(a) $y = \sin x$
The obvious example of a periodic function is $y = \sin x$, which goes through its complete range of values while x increases from $0°$ to $360°$. The period is therefore $360°$ or 2π radians and the amplitude, the maximum displacement from the position of rest, is 1.



(b) $y = 5 \sin 2x$
The amplitude is 5.
The period is $180°$ and there are thus 2 complete cycles in $360°$.



(c) $y = A \sin nx$

Thinking along the same lines, the function $y = A \sin nx$ has amplitude ....; period ....;
and will have .... complete cycles in 360°.

$$\text{amplitude} = A;\ \text{period} = \frac{360^\circ}{n} = \frac{2\pi}{n};\ n \text{ cycles in } 360^\circ$$

Graphs of $y = A \cos nx$ have the same characteristics.

By way of revising earlier work, then, complete the following short exercise.

Exercise

In each of the following, state (a) the amplitude and (b) the period.

1 $y = 3 \sin 5x$

2 $y = 2 \cos 3x$

3 $y = \sin \frac{x}{2}$

4 $y = 4 \sin 2x$

5 $y = 5 \cos 4x$

6 $y = 2 \sin x$

7 $y = 3 \cos 6x$

8 $y = 6 \sin \frac{2x}{3}$

Harmonics

A function $f(x)$ is sometimes expressed as a series of a number of different sine components. The component with the largest period is the *first harmonic*, or *fundamental* of $f(x)$.

$y = A_1 \sin x$	is the first harmonic or fundamental
$y = A_2 \sin 2x$	is the second harmonic
$y = A_3 \sin 3x$	is the third harmonic, etc.

and in general

$y = A_n \sin nx$ is the .... harmonic, with amplitude .... and period ....

$$n\text{th harmonic; amplitude } A_n;\ \text{period} = \frac{2\pi}{n}$$

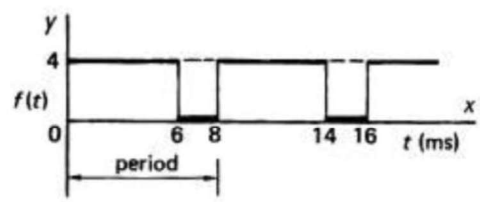
Non-sinusoidal periodic functions

Although we introduced the concept of a periodic function via a sine curve, a function can be periodic without being obviously sinusoidal in appearance.

Example

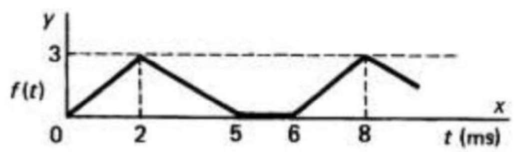
In the following cases, the x-axis carries a scale of t in milliseconds.

(a)



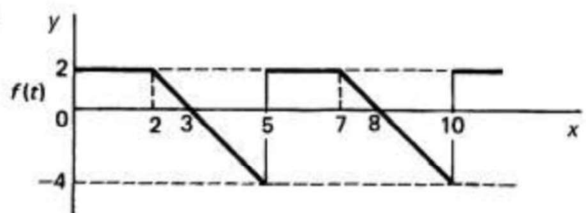
period = 8 ms

(b)



period = ...

(c)



period = ...

Orthogonal functions

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$ and

$$\int_a^b f(x)g(x)\,dx = 0$$

then we say that the two functions are **orthogonal** to each other on the interval $a \le x \le b$. In the previous frames we have seen that the trigonometric functions $\sin nx$ and $\cos nx$ where $n = 0, 1, 2, \dots$ form an infinite collection of periodic functions that are mutually orthogonal on the interval $-\pi \le x \le \pi$, indeed on any interval of width 2π. That is

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \quad \text{for } m \ne n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad \text{for } m \ne n$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0$$

Fourier series and Fourier Coefficient

Let f(x) be defined in the interval (-L,L) and outside of this interval by f(x+2L) =f(x) .i.e., f(x) is 2L-periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on(-L,L). The Fourier series or Fourier expansion corresponding to f(x) is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where the Fourier coefficients a_0, a_n and b_n are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad \text{For } n=1,2,3\dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Example: Determine the Fourier coefficient a_0,

Integrate both sides of the Fourier series (1), i.e.,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2}\,dx + \int_{-L}^{L} \sum_{n=1}^{\infty}\left\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right\} dx$$

Now $\int_{-L}^{L} \frac{a_0}{2}\,dx = a_0 L$, $\int_{-l}^{L} \sin\frac{n\pi x}{L}\,dx = 0$, $\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = 0$, therefore, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

Example

If the series $A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots,$

$$(a)\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad (b)\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad (c)\ A = \frac{a_0}{2}.$$

(*a*) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 13.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \qquad \text{if } m \neq 0\end{aligned}$$

Thus $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

Example: let us consider the function

$$f(x)=\begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Solution:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(-1)dx+\frac{1}{\pi}\int_{0}^{\pi}(1)dx$$

$$= -x|_{-\pi}^{0} + x|_0^{\pi} = 0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx=\frac{1}{\pi}\int_{-\pi}^{0}-\cos(nx)dx+\frac{1}{\pi}\int_{0}^{\pi}\cos(nx)dx$$

$$=\frac{1}{\pi}\left([-\frac{1}{n}\sin(nx)]_{-\pi}^{0}+[\frac{1}{n}\sin(nx)]_0^{\pi}\right)$$

$$=\frac{1}{n\pi}(\,-[\sin(0)-\sin(-\pi)]+[\sin(\pi)-\sin(0)]\,)$$

$$=0.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx=\frac{1}{\pi}\int_{-\pi}^{0}-\sin(nx)dx+\frac{1}{\pi}\int_{0}^{\pi}\sin(nx)dx$$

$$=\frac{1}{\pi}\left([\frac{1}{n}\cos(nx)]_{-\pi}^{0}+[-\frac{1}{n}\cos(nx)]_0^{\pi}\right)$$

$$=\frac{1}{n\pi}([\cos(0)-\cos(-n\pi)]+[\cos(0)-\cos(n\pi)])$$

$$=\frac{2}{n\pi}(1-\cos(n\pi))=\frac{2}{n\pi}(1-(-1)^n).$$

n	B
1	$\frac{4}{\pi}$
2	0
3	$\frac{4}{3\pi}$
4	0
5	$\frac{4}{5\pi}$

Thus the Fourier series of f is given as

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\sin(nx).$$

Example: Let us consider the function f defined as follows

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ 2-x, & 0 < x \le 2. \end{cases}$$

By using the formula of a_0, a_n and b_n, we find that

$$a_0 = \frac{1}{4}\int_0^2 (2-x)dx,$$

$$a_k = \frac{1}{2}\int_0^2 (2-x)\cos\frac{k\pi x}{2}dx, \; k = 1, 2, \dots$$

and

$$b_k = \frac{1}{2}\int_0^2 (2-x)\sin\frac{k\pi x}{2}dx, \; k = 1, 2, \dots$$

Evaluating these integrals gives

$$a_0 = \frac{1}{2}, \; a_k = \frac{2}{k^2\pi^2}[1-(-1)^k] \text{ and } b_k = \frac{2}{k\pi}$$

where use has been made of the fact that $\cos(k\pi) = (-1)^k$ and $\sin(k\pi) = 0$. Thus the Fourier series becomes

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{[1-(-1)^k]}{k^2\pi}\cos\frac{k\pi x}{2} + \frac{1}{k}\sin\frac{k\pi x}{2}\right).$$

Dirichlet Conditions

Suppose that

(1) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

(2) $f(x)$ is periodic outside $(-L, L)$ with period $2L$

(3) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series (*1*) with Fourier coefficients converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Example:

If the following functions are defined over the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$, state whether or not each function can be represented by a Fourier series.

1 $f(x) = x^3$

2 $f(x) = 4x - 5$

3 $f(x) = \dfrac{2}{x}$

4 $f(x) = \dfrac{1}{x-5}$

5 $f(x) = \tan x$

6 $f(x) = y$ where $x^2 + y^2 = 9$

Solution:

1 Yes

2 Yes

3 No: infinite discontinuity at $x = 0$

4 Yes

5 No: infinite discontinuity at $x = \pi/2$

6 No: two valued

Even and Odd Functions "Half-Range Expansions"

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval (0,L) which is half of the interval(-L,L) thus accounting for the name half range] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely,(-L,0). In such case, we have

$a_0=0$

$$a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for half range sine series}$$

$$b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for half range cosine series}$$

Fourier Sine series:

An *odd function* is a function with the property $f(-x) = -f(x)$.

For example :

1. $f(x) = x^3$. let $x = -1$, then $(-1)^3 = -(1)^3$

2. $f(x) = \sin(x)$. let $x = -\pi/2$, then $\sin(-\pi/2) = -\sin(\pi/2)$.

Note

1. The integral of an odd function over a symmetric interval is zero.
2. Since $a_n = 0$, all the cosine functions will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of Odd functions

Let us calculate the Fourier coefficients of an odd function:

$a_0=an=0$

but bn $\neq 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for half range sine series}$$

$$f(x) = \sum_{n=1}^{\infty} bn \sin\frac{n\pi x}{L}dx$$