Al-Mustaqbal University / College of Engineering & Technology
Communication Technical Engineering Department
Class (Second)
Signal and System / Code (UOMU0207032)
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1st term – Lecture 3. (Fourier Transform Properties)

5.6 PROPERTIES OF CONTINUOUS TIME FOURIER TRANSFORM

The Fourier transform has a number of important properties. These properties are useful for deriving Fourier transform pairs as well as for deducing general frequency domain relationships. These also help to find the effect of various time domain operations on the frequency domain. Some of the important properties are discussed as follows:

5.6.1 Linearity Property

The linearity property states that the Fourier transform of a weighted sum of two signals is equal to the weighted sum of their individual Fourier transforms.

i.e. If $x_1(t) \overset{FT}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{FT}{\longleftrightarrow} X_2(\omega)$

Then $ax_1(t) + bx_2(t) \overset{FT}{\longleftrightarrow} aX_1(\omega) + bX_2(\omega)$

where a and b are constants.

Proof: By definition,

$$F[ax_1(t) + bx_2(t)] = \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} ax_1(t)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} bx_2(t)\, e^{-j\omega t}\, dt$$

$$= a \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt + b \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$$

$$= aX_1(\omega) + bX_2(\omega)$$

$$\therefore \quad \boxed{ax_1(t) + bx_2(t) \overset{FT}{\longleftrightarrow} aX_1(\omega) + bX_2(\omega)}$$

5.6.2 Time Shifting Property

The time shifting property states that if a signal $x(t)$ is shifted by t_0 sec, the spectrum is modified by a linear phase shift of slope $-\omega t_0$, i.e.

If $x(t) \overset{FT}{\longleftrightarrow} X(\omega)$

Then $x(t - t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$

Proof: By definition,

$$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt$$

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Let
$$t - t_0 = p$$

$\therefore \quad t = p + t_0$ and $dt = dp$

$$\therefore \quad F[x(t - t_0)] = \int_{-\infty}^{\infty} x(p)\, e^{-j\omega(p + t_0)}\, dp$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(p)\, e^{-j\omega p}\, dp$$

$$= e^{-j\omega t_0} X(\omega)$$

$$\therefore \quad \boxed{x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)}$$

Similarly,
$$x(t + t_0) \xleftrightarrow{FT} e^{j\omega t_0} X(\omega)$$

This property has a very important implication. That is

$$|e^{-j\omega t_0} X(\omega)| = |X(\omega)|$$

and
$$\underline{|e^{-j\omega t_0} X(\omega)} = e^{-j\omega t_0} + \underline{|X(\omega)} = \underline{|-\omega t_0} + \underline{|X(\omega)}$$

This shows that shifting a function by t_0 results in multiplying its Fourier transform by $e^{-j\omega t_0}$. Thus, there is no change in magnitude spectrum but the phase spectrum is linearly shifted.

5.6.3 Frequency Shifting Property (Multiplication by an Exponential)

Frequency shifting property states that the multiplication of a time domain signal $x(t)$ by $e^{-j\omega t_0}$ results in the frequency spectrum shifted by ω_0, i.e.

If
$$x(t) \xleftrightarrow{FT} X(\omega)$$

Then
$$e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0)$$

Proof: By definition,

$$F[e^{j\omega_0 t} x(t)] = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0)t}\, dt$$

$$= X(\omega - \omega_0)$$

$$\therefore \quad \boxed{e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0)}$$

Similarly,
$$e^{-j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega + \omega_0)$$

5.6.4 Time Reversal Property

The time reversal property states that

If $$x(t) \xleftrightarrow{FT} X(\omega)$$

Then $$x(-t) \xleftrightarrow{FT} X(-\omega)$$

Proof: By definition,

$$F[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt$$

Replacing t by $-t$ in the RHS of the above expression for $F[x(-t)]$, we have

$$F[x(-t)] = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t}\, dt = X(-\omega)$$

$$\therefore \quad \boxed{x(-t) \xleftrightarrow{FT} X(-\omega)}$$

5.6.5 Time Scaling Property

Let $x(at)$ is a compressed version of $x(t)$ when $a > 1$ or expanded version of $x(t)$ when $a < 1$.

If $$x(t) \xleftrightarrow{FT} X(\omega)$$

Then $$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof: By definition,

$$F[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $$at = p$$

$$\therefore \quad t = \frac{p}{a} \quad \text{and} \quad dt = \frac{dp}{a}$$

$$F[x(at)] = \int_{-\infty}^{\infty} x(p)\, e^{-j\omega(p/a)}\, \frac{dp}{a}$$

$$\therefore \quad = \frac{1}{a} \int_{-\infty}^{\infty} x(p)\, e^{-j(\omega/a)p}\, dp$$

Case 1 When $a > 0$,

$$F[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(p)\, e^{-j(\omega/a)p}\, dp = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

CASE 2 When $a < 0$,

$$F[at] = \frac{1}{-a}\int_{-\infty}^{\infty} x(p)\, e^{j(\omega/a)p}\, dp = \frac{1}{-a}\int_{-\infty}^{\infty} x(p)\, e^{-j[-(\omega/a)]p}\, dp$$

$$= -\frac{1}{a} X\left(-\frac{\omega}{a}\right)$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$\therefore$

$$\boxed{x(at) \xleftrightarrow{\text{FT}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

5.6.6 Differentiation in Time Domain Property

The differentiation in time domain property states that the differentiation of a function in time domain is equivalent to the multiplication of its Fourier transform by a factor $j\omega$, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$\frac{d}{dt}x(t) \xleftrightarrow{\text{FT}} j\omega X(\omega)$$

Proof: By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Differentiating both sides w.r.t. t, we have

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}[e^{j\omega t}]\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

$$= j\omega\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= j\omega\, F^{-1}[X(\omega)]$$

$\therefore$

$$F\left[\frac{d}{dt}x(t)\right] = j\omega X(\omega)$$

$\therefore$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(\omega)}$$

In general, $\frac{d^n x(t)}{dt^n} \xleftrightarrow{FT} (j\omega)^n X(\omega)$

5.6.7 Differentiation in Frequency Domain Property

The differentiation in frequency domain property states that the multiplication of a signal $x(t)$ by t is equivalent to differentiation of its Fourier transform in frequency domain, i.e.

If $$x(t) \xleftrightarrow{FT} X(\omega)$$

Then $$tx(t) \xleftrightarrow{FT} j\frac{d}{d\omega} X(\omega)$$

Proof: By definition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Differentiating both sides w.r.t. ω, we have

$$\frac{d}{d\omega}[X(\omega)] = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}(e^{-j\omega t})\, dt = \int_{-\infty}^{\infty} x(t)(-jt)\, e^{-j\omega t}\, dt$$

$$= -j\int_{-\infty}^{\infty} [tx(t)]\, e^{-j\omega t}\, dt = -jF[tx(t)]$$

$\therefore$ $$F[tx(t)] = j\frac{d}{d\omega} X(\omega)$$

$\therefore$ $$\boxed{tx(t) \xleftrightarrow{FT} j\frac{d}{d\omega} X(\omega)}$$

5.6.8 Time Integration Property

The time integration property states that the integration of a function $x(t)$ in time domain is equivalent to the division of its Fourier transform by $j\omega$, i.e.

If $$x(t) \xleftrightarrow{FT} X(\omega)$$

Then $$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega), \text{ if } X(0) = 0$$

Proof: If $X(0) = 0$, this property can be easily proved by using integration by parts as in the case of the differentiation property.

By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Replacing t by a dummy variable τ, we have

$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega\tau}\,d\omega$$

Integrating both sides over $-\infty$ to t, we have

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega\tau}\,d\tau\right]d\omega$$

Interchanging the order of integration, we have

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(\int_{-\infty}^{t} e^{j\omega\tau}\,d\tau\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\frac{e^{j\omega\tau}}{j\omega}\right]_{-\infty}^{t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{X(\omega)}{j\omega}\right]e^{j\omega t}\,d\omega = \mathrm{F}^{-1}\left[\frac{X(\omega)}{j\omega}\right]$$

$\therefore$
$$\mathrm{F}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega}X(\omega)$$

$\therefore$
$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega}X(\omega)}$$

If $X(0) \neq 0$ then $x(t)$ is not an energy function, and the Fourier transform of $\int_{-\infty}^{t} x(\tau)\,d\tau$ includes an impulse function, that is

$$\mathrm{F}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega}X(\omega) + \pi X(0)\,\delta(\omega)$$

5.6.9 Convolution Property or Theorem

The convolution property or theorem states that the convolution of two signals in time domain is equivalent to the multiplication of their spectra in frequency domain. This is called the *time convolution theorem.*

If $\quad x_1(t) \xleftrightarrow{\text{FT}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\text{FT}} X_2(\omega)$

Then $\quad x_1(t) * x_2(t) \xleftrightarrow{\text{FT}} X_1(\omega)\,X_2(\omega)$

Proof: We know that the convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \qquad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Interchanging the order of integration, we have

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Substituting $t - \tau = p$ in the second integration, we have

$$t = p + \tau \quad \text{and} \quad dt = dp$$

$$\therefore \qquad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega(p+\tau)}\, dp \right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega p}\, dp \right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, X_2(\omega)\, e^{-j\omega\tau}\, d\tau$$

$$= \left[\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau \right] X_2(\omega) = X_1(\omega)\, X_2(\omega)$$

$$\therefore \qquad \boxed{x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)\, X_2(\omega)}$$

5.6.10 Multiplication Property or Theorem

The multiplication property or theorem states that the multiplication of two functions in time domain is equivalent to the convolution of their spectra in the frequency domain. This is called the frequency convolution theorem.

If $\qquad x_1(t) \xleftrightarrow{FT} X_1(\omega) \quad \text{and} \quad x_2(t) \xleftrightarrow{FT} X_2(\omega)$

Then $\qquad x_1(t)\, x_2(t) \xleftrightarrow{FT} \dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Proof: We know that

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and
$$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, e^{j\lambda t}\, d\lambda\right] x_2(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration, we get

$$F[x_1(t)\, x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{j\lambda t}\, e^{-j\omega t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j(\omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$\therefore$
$$\boxed{x_1(t)\, x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)}$$

or
$$2\pi x_1(t)\, x_2(t) \xleftrightarrow{FT} X_1(\omega) * X_2(\omega)$$

or
$$x_1(t)\, x_2(t) \xleftrightarrow{FT} X_1(f) * X_2(f)$$

5.6.11 Duality (Symmetry) Property

In spectrum analysis, the duality between the time and the frequency is exhibited. The duality (symmetry) property states that

If
$$x(t) \xleftrightarrow{FT} X(\omega)$$

Then
$$X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$$

Proof: By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$\therefore$ $$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

or $$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Interchanging t and ω, we have

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt = \text{F}[X(t)]$$

$\therefore$ $$\text{F}[X(t)] = 2\pi x(-\omega)$$

i.e. $$\boxed{X(t) \xleftrightarrow{\text{FT}} 2\pi x(-\omega)}$$

For even functions,

$$x(-\omega) = x(\omega)$$

$\therefore$ $$X(t) \xleftrightarrow{\text{FT}} 2\pi x(\omega)$$

5.6.12 Modulation Property

The modulation property states that, if a signal $x(t)$ is multiplied by $\cos \omega_c t$, its spectrum gets translated up and down in frequency by ω_c, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$x(t) \cos \omega_c t \xleftrightarrow{\text{FT}} \frac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]$$

Proof: $$x(t) \cos \omega_c t = x(t) \left[\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right]$$

$\therefore$ $$\text{F}[x(t) \cos \omega_c t] = \text{F}\left[\frac{x(t)}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]\right]$$

$$= \frac{1}{2}\text{F}[x(t)\, e^{j\omega_c t}] + \frac{1}{2}\text{F}[x(t)\, e^{-j\omega_c t}] = \frac{1}{2}X[\omega - \omega_c] + \frac{1}{2}X[\omega + \omega_c]$$

$$= \frac{1}{2}[X[\omega - \omega_c] + X[\omega + \omega_c]]$$

$\therefore$ $$\boxed{x(t) \cos \omega_c t \xleftrightarrow{\text{FT}} \frac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]}$$

Similarly, $$x(t) \sin \omega_c t \xleftrightarrow{\text{FT}} \frac{1}{2j}[X(\omega - \omega_c) - X(\omega + \omega_c)]$$