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1ST TERM (THE CARTESIAN COORDINATE SYSTEM THE VECTOR FIELD - DOT PRODUCT - CROSS PRODUCT)(2)
ELECTROMAGNETIC STATIC FIELDS/(CODE)(UOMU0207035)

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THE CARTESIAN COORDINATE SYSTEMS

2.1 Introduction

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or nonorthogonal.

An **orthogonal** system is one in which the coordinates are mutually perpendicular

2.2 The Cartesian Coordinate System (x, y, z)

In the Cartesian coordinate system we set up three coordinate axes mutually at right angles to each other, and call them the x, y, and z-axes. It is customary to choose a right-handed coordinate system, in which a rotation (through the smaller angle) of the x-axis into the y-axis would cause a right-handed screw to progress in the direction of the z-axis. Figure 1. shows a right-handed Cartesian coordinate system.

A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of a perpendicular dropped from the point to the x, y, and z-axes.

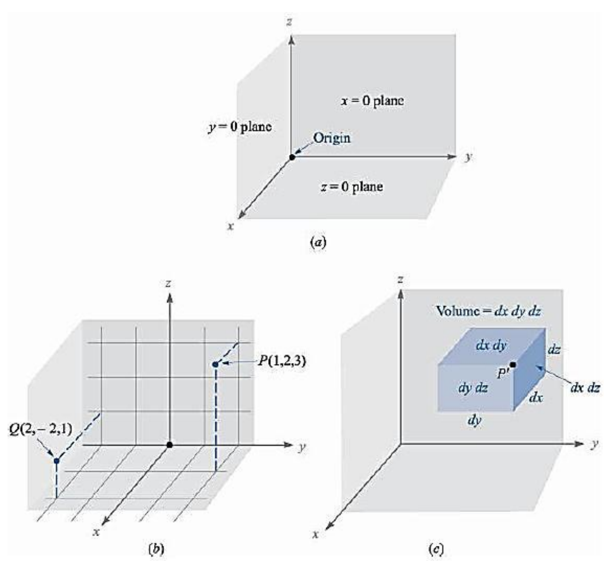

Figure 1(a) A right-handed Cartesian coordinate system. (b) The location of points P(1, 2, 3) and Q(2, -2, 1). (c) The differential volume element in Cartesian coordinates

Also shown in Figure 1 (c) are differential element in length, area, and volume. Notes from the figure that in Cartesian coordinate:

1. **Differential displacement is given by**

$$d\vec{L} = dx\boldsymbol{a}_x + dy\boldsymbol{a}_y + dz\boldsymbol{a}_z$$

2. **Differential normal area is given by**

$$d\bar{S} = dydz\ \boldsymbol{a}_x$$

$$d\vec{S} = dxdz\ \boldsymbol{a}_y$$

$$d\vec{S} = dxdy\ \boldsymbol{a}_z$$

3. **Differential volume is given by**

$$dV = dx\ dy\ dz$$

The ranges of the coordinate variables x, y, and z are

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

A vector $\vec{A}$ in Cartesian coordinates can be written as shown in Figure

$$\vec{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$$

Where a_x , a_y, and a_z are unit vectors along the x, y, and z-directions.

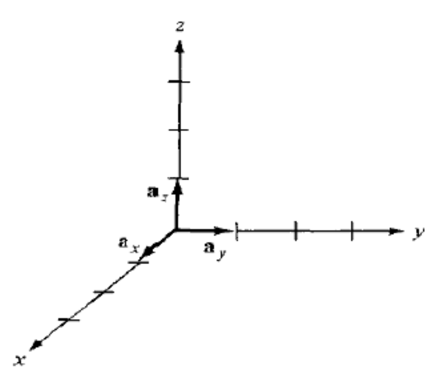

Figure 2 Unit vectors a_x ,a_y ,and a_z

VECTOR MULTIPLICATION

When two vectors A and B are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

1. Scalar (or dot) product: $\mathbf{A} \cdot \mathbf{B}$
2. Vector (or cross) product: $\mathbf{A} \times \mathbf{B}$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

3. Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

or

4. Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

A. <u>The Dot Product</u>

The dot product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \cdot \vec{B}$, is defined geometrically as the dot product of the magnitude of $\vec{B}$ and the projection of $\vec{A}$ onto $\vec{B}$ (or vice versa)

Thus:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta_{AB}$$

where $\boldsymbol{\theta_{AB}}$ is the smaller angle between **A** and **B**, the result of $\mathbf{A} \cdot \mathbf{B}$ is called either *scalar product* since it is scalar, or *dot product* due to sign.

If $\vec{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z$ and $\vec{B} = B_x \boldsymbol{a}_x + B_y \boldsymbol{a}_y + B_z \boldsymbol{a}_z$ then

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Not that

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_y = \boldsymbol{a}_y \cdot \boldsymbol{a}_z = \boldsymbol{a}_z \cdot \boldsymbol{a}_x = 0 \qquad (\theta = 90, \quad \cos\theta, \quad \cos 90 = 0)$$

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_x = \boldsymbol{a}_y \cdot \boldsymbol{a}_y = \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1 \qquad (\theta = 0, \quad \cos\theta, \quad \cos 0 = 1)$$

Example: The three vertices of a triangle are located at $A(6,-1,2)$, $B(-2,3,-4)$ and C(-3, 1, 5). Find: (a) $\vec{R}_{AB}$; (b) $\vec{R}_{AC}$; (c) the angle θ_{BAC} at vertex A ?

Solution:

(a) $\vec{R}_{AB} = (-2,3,-4) - (6,-1,2)$

$$= (-2-6)\boldsymbol{a}_x + \big(3-(-1)\big)\boldsymbol{a}_y + (-4-2\big)\boldsymbol{a}_z$$

$$= -8\boldsymbol{a}_x + 4\boldsymbol{a}_y - 6\boldsymbol{a}_z$$

(b) $\vec{R}_{AC} = (-3,1,5) - (6,-1,2)$

$$= (-3-6)\boldsymbol{a}_x + \big(1-(-1)\big)\boldsymbol{a}_y + (5-2\big)\boldsymbol{a}_z$$

$$= -9\boldsymbol{a}_x + 2\boldsymbol{a}_y + 3\boldsymbol{a}_z$$

(c) $\left|\vec{R}_{AB}\right| = \sqrt{8^2+4^2+6^2} = 10.77$

$$\left|\vec{R}_{AC}\right| = \sqrt{9^2+2+3^2} = 9.69$$

$$\vec{R}_{AB}\cdot\vec{R}_{AC} = A_xB_x + A_yB_y + A_zB_z$$

$$= -8(-9) + 4(2) - 6(3) = 62$$

$$\vec{R}_{AB}\cdot\vec{R}_{AC} = \left|\vec{R}_{AB}\right|\left|\vec{R}_{AC}\right| \cos\theta_{BAC}$$

$$62 = 10.77(9.69)\cos\theta_{BAC}$$

$$\cos\theta_{BAC} = \frac{62}{104.36}$$

$$\theta_{BAC} = \cos^{-1} 0.594$$

$$\therefore \theta_{BAC} = 53.55^{\circ}$$

B. Cross Product

The cross product of two vectors **A** and **B**, written as $\vec{A} \times \vec{B}$, is defined as

$$\vec{A} \times \vec{B} = AB \sin \theta_{AB} \, \boldsymbol{a}_n$$

Where is a unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$. The direction of is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Fig. 1.3(a). Alternatively, the direction of $\boldsymbol{a}_n$ is taken as that of the advance of a right-handed screw as **A** is turned into **B** as shown in Fig.

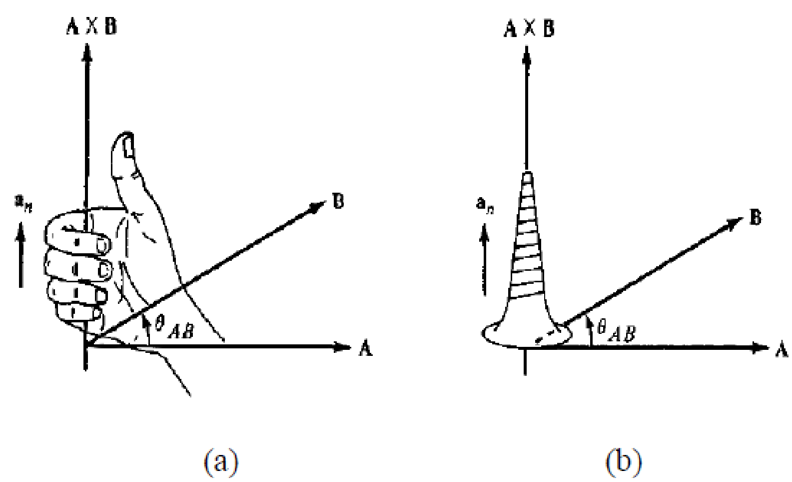

Figure 1.3: Direction of $\vec{A} \times \vec{B}$ and $\boldsymbol{a}_n$ using (a) right hand rule (b) a right-handed screw.

The vector multiplication of equation above is called *cross product* due to the cross sign; it is also called *vector product* since the result is a vector.

If $\vec{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$ and $\vec{B} = B_x\boldsymbol{a}_x + B_y\boldsymbol{a}_y + B_z\boldsymbol{a}_z$ then

$$\vec{A} \times \vec{B} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ A_{\boldsymbol{x}} & A_{\boldsymbol{y}} & A_{\boldsymbol{z}} \\ B_{\boldsymbol{x}} & B_{\boldsymbol{y}} & B_{\boldsymbol{z}} \end{bmatrix}$$

$$= (A_{\boldsymbol{y}} B_z - A_{\boldsymbol{z}} B_y)\boldsymbol{a}_x - (A_{\boldsymbol{x}} B_z - A_{\boldsymbol{z}} B_x)\boldsymbol{a}_y + (A_{\boldsymbol{x}} B_y - A_{\boldsymbol{y}} B_x)\boldsymbol{a}_z$$

Note that

$\boldsymbol{a}_x \times \boldsymbol{a}_x = \boldsymbol{a}_y \times \boldsymbol{a}_y = \boldsymbol{a}_z \times \boldsymbol{a}_z = 0$ $\qquad \theta = 0, \quad \sin\theta = \sin 0 = 0$

$\boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$

$\boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$

$\boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$ while $\boldsymbol{a}_y \times \boldsymbol{a}_x = -\boldsymbol{a}_z$

Example: If three points $A(6,-1,2)$, $B(-2,3,-4)$ and $C(-3,1,5)$.

Find $\vec{R}_{AB} \times \vec{R}_{AC}$

Solution:

$$\vec{R}_{AB} = (-2,3,-4) - (6,-1,2)$$
$$= (-2-6)\boldsymbol{a}_x + \big(3-(-1)\big)\boldsymbol{a}_y + (-4-2)\boldsymbol{a}_z$$
$$= -8\boldsymbol{a}_x + 4\boldsymbol{a}_y - 6\boldsymbol{a}_z$$

$$\vec{R}_{AC} = (-3,1,5) - (6,-1,2)$$
$$= (-3-6)\boldsymbol{a}_x + \big(1-(-1)\big)\boldsymbol{a}_y + (5-2)\boldsymbol{a}_z$$
$$= -9\boldsymbol{a}_x + 2\boldsymbol{a}_y + 3\boldsymbol{a}_z$$

$$|\vec{R}_{AB}| = \sqrt{8^2 + 4^2 + 6^2} = 10.77$$

$$|\vec{R}_{AC}| = \sqrt{9^2 + 2^2 + 3^2} = 9.69$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \mathrm{R}_{AB_x} & \mathrm{R}_{AB_y} & \mathrm{R}_{AB_z} \\ \mathrm{R}_{AC_x} & \mathrm{R}_{AC_y} & \mathrm{R}_{AC_z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ -8 & 4 & -6 \\ -9 & 2 & 3 \end{bmatrix}$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = [4*3 - (-6*2)]\boldsymbol{a}_x - [(-8*3) - (-6*-9)]\boldsymbol{a}_y + (-8*2 - 4(-9))\boldsymbol{a}_z$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = 24\boldsymbol{a}_x + 78\boldsymbol{a}_y + 20\boldsymbol{a}_z$$

$$|\vec{R}_{AB} \times \vec{R}_{AC}| = \sqrt{24^2 + 78^2 + 20^2} = |\vec{R}_{AB}||\vec{R}_{AC}| \sin\theta_{BAC}$$

$$\theta_{BAC} = \sin^{-1}\frac{|\vec{R}_{AB}\times\vec{R}_{AC}|}{|\vec{R}_{AB}||\vec{R}_{AC}|} = \sin^{-1}\frac{\sqrt{24^2+78^2+20^2}}{10.77*9.69} \qquad \therefore \theta_{BAC} = 53.6^\circ$$

Example: Given vectors $\mathbf{A} = 3\boldsymbol{a}_x + 4\boldsymbol{a}_y + \boldsymbol{a}_z$ and $\mathbf{B} = 2\boldsymbol{a}_y - 5\boldsymbol{a}_z$, find the angle between A and B.

Solution:

The angle θ_{AB} can be found by using either dot product or cross product.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta_{AB}$$

$$\vec{A} \cdot \vec{B} = (3, 4, 1) \cdot (0, 2, -5)$$

$$= 0 + 8 - 5 = 3$$

$$|\vec{A}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$

$$|\vec{B}| = \sqrt{0^2 + 2^2 + 5^2} = \sqrt{29}$$

$$\cos\theta_{AB} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \frac{3}{\sqrt{(26)(29)}} = 0.1092$$

$$\theta_{AB} = \cos^{-1} 0.1092 = 83.73°$$

Alternatively:

$$\vec{A} \times \vec{B} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{bmatrix}$$

$$= (-20 - 2)\boldsymbol{a}_x - (-15 - 0)\boldsymbol{a}_y + (6 - 0)\boldsymbol{a}_z$$

$$= -22\boldsymbol{a}_x + 15\boldsymbol{a}_y + 6\boldsymbol{a}_z$$

$$|\vec{A} \times \vec{B}| = \sqrt{(-22)^2 + 15^2 + 6^2} = \sqrt{745}$$

$$\sin\theta_{AB} = \frac{\vec{A} \times \vec{B}}{|\vec{A}||\vec{B}|} = \frac{\sqrt{745}}{\sqrt{(26)(29)}} = 0.994$$

$$\theta_{AB} = \sin^{-1} 0.994 = 83.73°$$