Applications of Gauss's Law

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a *Gaussian surface*). The surface over which Gauss's law is applied must be closed, but it can be made up of several surface elements. Thus the defining

conditions of a special Gaussian surface are

a- The surface is closed.

b- At each point of the surface $\vec{D}$ is either normal or tangential to the surface, so that

($\vec{D}s$. d$\vec{S}$) becomes either ($\vec{D}$s d$\vec{S}$) or (zero), respectively

c- $\vec{D}$ is sectional constant over that part of the surface where $\vec{D}$ is normal.

Symmetrical Charge Distributions

1. **Point Charge.**

Suppose a point charge Q is located at the origin. To determine D at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, *a spherical surface* centered at the origin is the *Gaussian surface* in this case and is shown in Figure 1

$$\Psi = \oint_S \vec{D}_S \cdot d\vec{S} = D_r \oint d\vec{S}$$

$$\Psi = D_r \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \; d\theta \; d\emptyset = 4\pi \, r^2 \, D_r$$

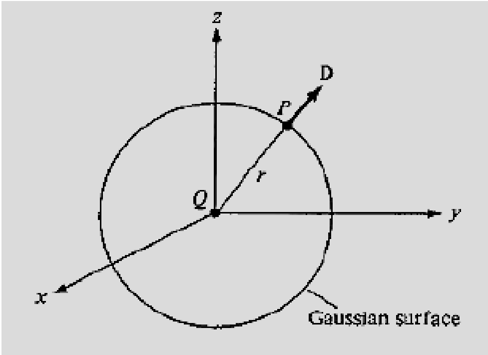


Figure 1 Gaussian surface about a point charge

$$Q_{enclosed} = Q$$

$$\Psi = Q_{enclosed}$$

$$4\pi r^2 D_r = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2} \, \boldsymbol{a}_r$$

2. Infinite Line Charge

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine D at a point P, we choose a *cylindrical surface* containing P to satisfy symmetry condition as shown in

Figure 2 . D is constant on and normal to the cylindrical Gaussian surface; i.e., =Dρ aρ. If we apply Gauss's law to an arbitrary length L of the line

$$\Psi = \int \vec{D}_S \cdot d\vec{S} = D_\rho \int d\vec{S}$$

$$\Psi = \vec{D}_\rho \int_{z=0}^{l} \int_{\emptyset=0}^{2\pi} \rho \, d\emptyset \, dz$$

$$\Psi = 2\pi \, \rho \, l \, D_\rho$$

$$Q_{enclosed} = \int \rho_L \, dl$$

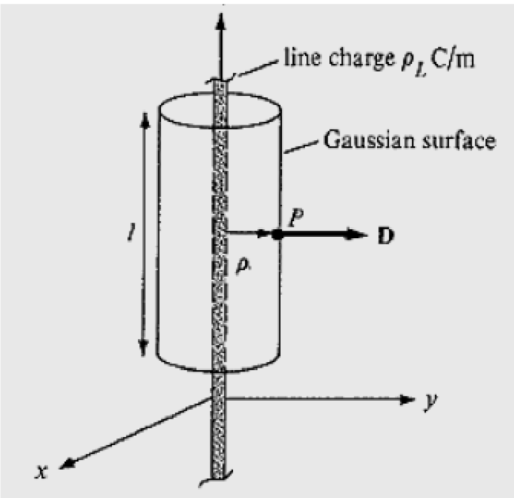


Figure 2. Gaussian surface about an infinite line

$$Q_{enclosed} = \rho_L \int_{z=0}^{l} dz = l\,\rho_L$$

$$\Psi = Q_{enclosed}$$

$$2\pi\,\rho\,l\,D_\rho = l\,\rho_L$$

$$D_\rho = \frac{\rho_L}{2\pi\,\rho}$$

$$\vec{D} = \frac{\rho_L}{2\pi\,\rho}\boldsymbol{a}_\rho$$

3. **<u>Uniformly Charged Sphere</u>**

Consider a sphere of radius a with a uniform charge ρ_v C/m^3. To determine D everywhere, we construct Gaussian surfaces for cases r < a, and r > a separately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

For r < a, the total charge enclosed by the spherical surface of radius r, as shown in Figure 3.(a), is

$$\Psi = \oint \vec{D}_S \cdot d\vec{S} = \vec{D}_r \oint d\vec{S} = \vec{D}_r \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta d\emptyset = 4\pi r^2 \vec{D}_r$$

$$Q_{enclosed} = \int_{vol} \rho_v \, dv = \rho_v \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{0}^{r} r^2 \sin\theta \, d\theta d\emptyset \, dr = \frac{4}{3} \rho_v \pi r^3$$

$$\Psi = Q_{enclosed} \qquad (Gauss's\ Law)$$

$$4\pi r^2 \vec{D}_r = \frac{4}{3} \rho_v \pi r^3$$

$$\vec{D}_r = \frac{\rho_v}{3} r$$

$$\vec{D} = \frac{\rho_v}{3} r \, \boldsymbol{a}_r \qquad (r < a)$$

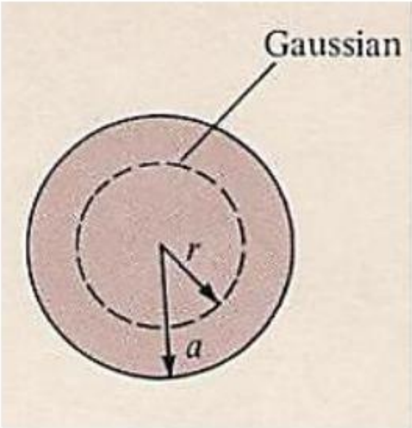


Figure 3.(a)

For r ≥ a, the Gaussian surface is shown in Figure 3.(b). The charge enclosed by the surface is the entire charge in this case, i.e.,

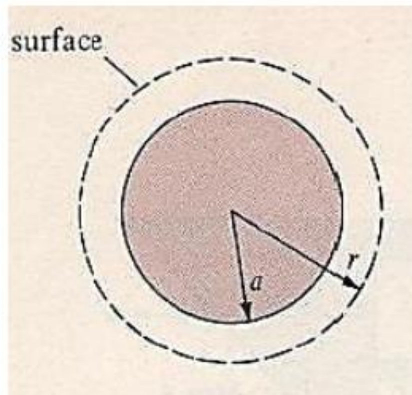
$$Q_{enclosed} = \int_{vol} \rho_v \, dv = \rho_v \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{0}^{a} r^2 \sin\theta \, d\theta \, d\emptyset \, dr = \frac{4}{3} \rho_v \, \pi a^3$$

$$\Psi = Q_{enclosed} \qquad (Gauss's\ Law)$$

$$4\pi r^2 \vec{D}_r = \frac{4}{3} \rho_v \, \pi a^3$$

$$\vec{D}_r = \frac{a^3}{3\, r^2} \rho_v$$

$$\vec{D} = \frac{a^3}{3\, r^2} \rho_v \; \boldsymbol{a}_r \qquad (r \geq a)$$

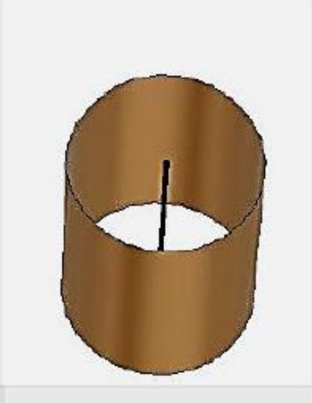


Example: A uniform line charge of $\rho_L = 3\mu$ C/m lies along the z axis, and a concentric circular cylinder of radius 2 m has $\rho_s = \frac{-1.5}{4\pi}$ μC/m^2. Both distributions are infinite in extent with z. Use Gauss's law to find D in all regions?

Solution:

$1 - The\ region\ 0 < \rho < 2$

Using Gaussian surface cylinder ρ



$$\Psi = Q_{enclosed}$$

$$\Psi = D_\rho \int_{z=0}^{l} \int_{\emptyset=0}^{2\pi} \rho d\emptyset dz = 2\pi\rho L D_\rho$$

$$Q_{enclosed} = \int \rho_L dL = \rho_L \int_{z=0}^{L} d_z = L\,\rho_L$$

$$2\pi\rho L D_\rho = L\,\rho_L$$

$$D = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho = \frac{3\times10^{-6}}{2\pi\rho}\mathbf{a}_\rho = \frac{0.477}{\rho}\mathbf{a}_\rho \quad \mu C/m^2$$

$2 - \text{The region } 2 < \rho$

$Q_{enclosed} = Q_1 + Q_2$

$Q_1 = L\,\rho_L$

$Q_2 = \int \rho_s\, dS$

$$Q_2 = \rho_s \int_0^L \int_0^{2\pi} \rho\, d\emptyset\, dz = 2 \times 2\pi \times L \times \rho_s = 4\pi L \rho_s$$

$Q_{enclosed} = Q_1 + Q_2 = L\,\rho_L + 4\pi L \rho_s$

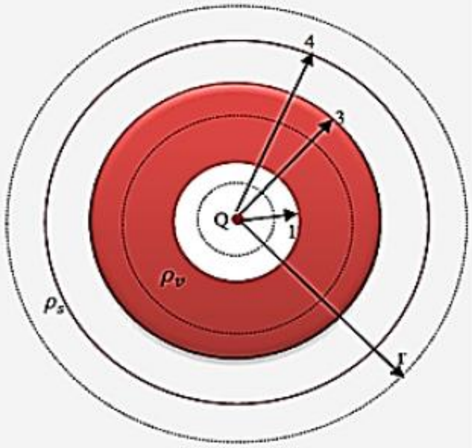
$\Psi = Q_{enclosed}$

$2\pi\rho L D_\rho = L\,\rho_L + 4\pi L \rho_s$

$$D = \frac{\rho_L + 4\pi\rho_s}{2\pi\rho}\,\mathbf{a}_\rho = \frac{0.239}{\rho}\mathbf{a}_\rho \quad \mu C/m^2$$

Example: A point charge Q=2π c at the origin , a volume charge density of 4 c/m³ at the region $1 \le r \le 3$ and a sheet of charge -6 c/m² has $r = 4$, Find D at $r = 0.5$, $r = 2$, $r = 5$?

Solution:



Gaussian surface

D at $r = 0.5$

D is due to a point charge

$$D = \frac{Q}{4\pi r^2}\,\mathbf{a}_r = \frac{2\pi}{4\pi(0.5)^2}\,\mathbf{a}_r = 2\,\mathbf{a}_r$$

D at $r = 2$

$$\Psi = Q_{enclosed}$$

$$\Psi = 4\pi r^2 D_r$$

$$Q_{enclosed} = Q_1 + Q_2$$

$$Q_2 = \int_{vol} \rho_v\, d_v = 4\int_0^{2\pi}\int_0^{\pi}\int_1^{2} r^2 \sin\theta\, dr d\theta d\emptyset = 4 * 2 * 2\pi * \left[\frac{r^3}{3}\right]_1^2 = \frac{112\pi}{3}$$

$$\therefore 4\pi r^2 D_r = 2\pi + \frac{112\pi}{3}$$

$$D = \frac{39.33}{4(2)^2}\,\mathbf{a}_r = 2.45\,\mathbf{a}_r$$

$\boldsymbol{D}$ at $\boldsymbol{r} = 5$

$$Q_{enclosed} = Q_1 + Q_2 + Q_3$$

$$Q_2 = 4 * 2 * 2\pi * \left[\frac{r^3}{3}\right]_1^3 = \frac{416\pi}{3}$$

$$Q_3 = \int \rho_s\, dS = -6\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta d\emptyset = -6 * (4)^2 * 2\pi * 2 = -384\pi$$

$$\therefore 4\pi r^2 D_r = 2\pi + \frac{416\pi}{3} - 384\pi = -243.33\pi$$

$$D_r = \frac{-243.33\pi}{4\pi r^2}\,\mathbf{a}_r = \frac{-243.33}{4(5)^2}\,\mathbf{a}_r = -2.433\,\mathbf{a}_r$$