Divergence (div)

There are two main indicators of the manner in which a vector field changes from point to point throughout space. The first of these is divergence, it is a scalar and bears a similarity to the derivative of a function. The second is curl.

When the divergence of a vector field is nonzero, that region is said to contain sources or sinks, sources when the divergence is positive, sinks when negative. In static electric fields there is a correspondence between positive divergence, sources, and positive electric charge Q. Electric flux Ψ by definition originates on positive charge. Thus, a region which contains positive charges contains the sources of Ψ. The divergence of the electric flux density D will be positive in this region. A similar correspondence exists between negative divergence, sinks, and negative electric charge

$$\text{Divergence of A} = \text{div A} = \lim_{\Delta v \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{(Cartesian)}$$

$$\text{div D} = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div D} = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

The Divergence Theorem

Gauss' law states that the closed surface integral of D.dS is equal to the charge enclosed. If the charge density function ρv is known throughout the volume, then the charge enclosed may be obtained from an integration ρv of throughout the volume. Thus

$$\oint D_S \cdot dS = \int_{vol} \rho_v dv$$

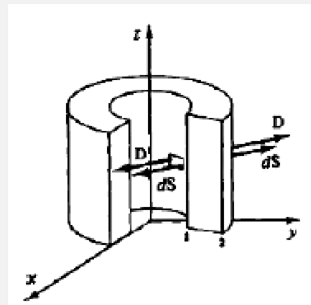
But $\nabla . \mathrm{D} = \rho_v$ and so

$$\oint D_S \cdot dS = \int_{vol} (\nabla . \mathrm{D}) dv$$

This is the divergence theorem, also known as Gauss' divergence theorem. Of course, the volume v is that which is enclosed by the surface S.

Example: Given that $D = (10\rho^3/4)\ a\rho$, (C/m^2) in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by ρ= 1m, ρ= 2m, z = 0 and z = 10

Solution:



The left side of divergence theorem is:

$$\oint D_S \,.dS = \int_0^{10}\int_0^{2\pi} \frac{10\rho^3}{4}\rho d\emptyset dz = \frac{10}{4}\times \rho^4 \times 2\pi \times 10$$

$$= 50\pi\ \rho^4 = 50\pi(2^4 - 1^4) = 750\pi$$

The right side is:

$$\int_{vol}(\nabla.\mathrm{D})dv = \int_{vol}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho D_\rho\right)dv = \int_{vol}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{10\rho^3}{4}\right)\right)dv$$

$$= \int_{vol}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{10\rho^4}{4}\right)\right)dv = \int_{vol}(10\rho^2)dv = \int_0^{10}\int_0^{2\pi}\int_1^{2} 10\rho^2\ \rho\ d\rho\ d\emptyset\ dz$$

$$= 10\times 2\pi\times 10\times\left[\frac{\rho^4}{4}\right]_1^2 = 750\pi$$

Maxwell's First Equation (Electrostatics)

Maxwell's first equation, which describes the electrostatic field, is derived immediately from Gauss's theorem, which in turn is a consequence of Coulomb's inverse square law. Gauss's theorem states that the surface integral of the electrostatic field

over a closed surface is equal to the charge enclosed by that surface. That is

$$\int_{\text{surface}} \mathbf{D}\cdot d\boldsymbol{\sigma} = \int_{\text{volume}} \rho\, dv$$

Here ρ is the charge per unit volume.

But the surface integral of a vector field over a closed surface is equal to the volume integral of its divergence, and therefore

$$\int_{\text{surface}} \operatorname{div} \mathbf{D}\, dv = \int_{\text{volume}} \rho\, dv$$

Therefore

$$\operatorname{div}\mathbf{D} = \rho,$$

or, in the nabla notation,

$$\nabla \cdot \mathbf{D} = \rho.$$

This is the first of Maxwell's equations.

We now wish to consolidate the gains of the last two sections and to provide an interpretation of the divergence operation as it relates to electric flux density. The expressions developed there may be written as

$$\operatorname{div} \mathrm{D} = \lim_{\Delta v \to 0} \frac{\oint D_S \,.dS}{\Delta v}$$

$$\operatorname{div} \mathrm{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\operatorname{div} \mathrm{D} = \rho_v$$

$$\nabla . \mathrm{D} = \rho_v$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that ***the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there***. This equation is called the point form of **Gauss's law**. Gauss's law relates the flux leaving any closed surface to the charge enclosed,

Example: in the region $a \le \rho \le b$, $\vec{\mathbf{D}} = \rho_0\left(\frac{\rho^2-a^2}{2\rho}\right)\mathbf{a}_\rho$,

and for $\rho > b$ $\vec{\mathbf{D}} = \rho_0\left(\frac{b^2-a^2}{2\rho}\right)\mathbf{a}_\rho$

for $\rho < a$ $\vec{\mathbf{D}} = \mathbf{0}$, find ρ_v in all three regions?

Solution:

$1 - for\ the\ region\ a \le \rho \le b$

$$D_\rho = \rho_0\left(\frac{\rho^2-a^2}{2\rho}\right) \quad , \quad D_\emptyset = 0 \quad , \quad D_z = 0$$

$$\nabla . \mathrm{D} = \rho_v$$

$$\rho_v = \nabla . \mathrm{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\emptyset}{\partial \emptyset} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\left(\rho_0\left(\frac{\rho^2-a^2}{2\rho}\right)\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho_0\left(\frac{\rho^2-a^2}{2}\right) = \frac{\rho_0}{2\rho}\frac{\partial}{\partial \rho}(\rho^2-a^2) = \frac{\rho_0}{2\rho}\times 2\rho = \rho_0\ C/m^3$$

$2-$ *for the region* $\rho > b$

$$\rho_v = \nabla.\mathrm{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho D_\rho = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\rho_0\left(\frac{b^2-a^2}{2\rho}\right)\right) == \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho_0\left(\frac{b^2-a^2}{2}\right) = \frac{\rho_0}{2\rho}\frac{\partial}{\partial\rho}(b^2-a^2) = 0$$

$3-$ *for the region* $\rho < a$

$$\rho_v = \nabla.\mathrm{D} = \rho_v = \nabla.0 = 0$$

Gradient

The vector field ∇V (also written grad V) is called the gradient of the scalar function V

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(Cartesian)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \emptyset}\mathbf{a}_\emptyset + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \emptyset}\mathbf{a}_\emptyset \qquad \text{(spherical)}$$

CURL OF A VECTOR

The curl of $\vec{A}$ is a rotational vector whose magnitude is the maximum circulation of A ⃗ per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$curl\ \vec{A} = \vec{\nabla} \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{L}}{\Delta S} \right) \boldsymbol{a}_{n_{max}}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or

$$\vec{\nabla} \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] \boldsymbol{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] \boldsymbol{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \boldsymbol{a}_z \quad \textbf{Cartesian}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \boldsymbol{a}_\rho & \rho \boldsymbol{a}_\emptyset & \boldsymbol{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\emptyset & A_z \end{vmatrix}$$

or

$$\vec{\nabla} \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \emptyset} - \frac{\partial A_\emptyset}{\partial z}\right] \boldsymbol{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] \boldsymbol{a}_\emptyset + \frac{1}{\boldsymbol{\rho}}\left[\frac{\partial (\rho A_\emptyset)}{\partial \rho} - \frac{\partial A_\rho}{\partial \emptyset}\right] \boldsymbol{a}_z \ \textbf{Cylindrical}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{a}_r & r\boldsymbol{a}_\theta & r\, sin\theta\, \boldsymbol{a}_\emptyset \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_r & rA_\theta & r\, sin\theta\, A_\emptyset \end{vmatrix}$$

or

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\emptyset \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\emptyset}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\emptyset} - \frac{\partial(rA_\emptyset)}{\partial r}\right]\boldsymbol{a}_\theta$$

$$+\frac{1}{\boldsymbol{r}}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\boldsymbol{a}_\emptyset \qquad \textbf{\textit{spherical}}$$

Frequently useful are two properties of the curl operator:

1) If the divergence of a curl is zero; $(\nabla \times \vec{A}) = 0$*, then the vector field* $\vec{A}$ *is the Electric Field.*

2) The curl of a gradient is the zero vector; $(\nabla \times (\nabla\vec{A}) = 0$