Mechanics — Lecture 6

Motion in Three Dimensions, Potential Energy,and the Harmonic Oscillator

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1 Motion of a Particle in Three Dimensions

In many realistic physical systems, particles are free to move in space rather than being constrained to a straight line. Examples include electrons in electromagnetic fields, molecules in gases and liquids, and particles in biological or medical environments. To accurately describe such motion, a three-dimensional framework based on vector quantities is required.

This section introduces the mathematical description of motion in three dimensions using position, velocity, and acceleration vectors. These quantities form the foundation for analyzing more complex dynamical behavior.

A particle moving in three dimensions is described using vector quantities. The position vector is

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

Velocity and acceleration are obtained by time differentiation.

Solved Problem 1: Velocity and Acceleration

Problem: A particle moves such that

$$\vec{r}(t) = (2t^3 - t)\hat{i} + (t^2 + 4t)\hat{j} + (5 - t^2)\hat{k}.$$

Find: (a) the velocity vector, (b) the acceleration vector, (c) the magnitude of the velocity at $t = 1$ s.

Solution:

Velocity:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (6t^2 - 1)\hat{i} + (2t + 4)\hat{j} - 2t\hat{k}.$$

Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = 12t\hat{i} + 2\hat{j} - 2\hat{k}.$$

At $t = 1$:

$$\vec{v}(1) = 5\hat{i} + 6\hat{j} - 2\hat{k}.$$

Magnitude:

$$|\vec{v}| = \sqrt{5^2 + 6^2 + (-2)^2} = \sqrt{65}.$$

Solved Problem 2: Trajectory Elimination

Problem: A particle moves according to

$$x = 2t, \qquad y = t^2, \qquad z = 4t.$$

Eliminate t and find the equation of the trajectory.

Solution:

From $x = 2t$:

$$t = \frac{x}{2}.$$

Substitute into y and z:

$$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}, \qquad z = 4\left(\frac{x}{2}\right) = 2x.$$

The trajectory is a **parabolic curve** in three-dimensional space.

Solved Problem 3: Force from Motion

Problem: A particle of mass 3 kg has position

$$\vec{r}(t) = (t^2)\hat{i} + (2t)\hat{j} + (t^3)\hat{k}.$$

Find the force acting on the particle at $t = 2$ s.

Solution:

Velocity:

$$\vec{v} = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}.$$

Acceleration:

$$\vec{a} = 2\hat{i} + 6t\hat{k}.$$

At $t = 2$:

$$\vec{a}(2) = 2\hat{i} + 12\hat{k}.$$

Force:

$$\vec{F} = m\vec{a} = 3(2\hat{i} + 12\hat{k}) = 6\hat{i} + 36\hat{k}.$$

2 Potential Energy and the Del Operator

The motion of a particle is specified by its position vector as a function of time. Velocity and acceleration are obtained by differentiating the position vector with respect to time. For conservative systems, force is related to potential energy by

$$\vec{F} = -\nabla U.$$

Solved Problem 4: Gradient and Force

Problem: Given the potential energy

$$U(x, y, z) = 3x^2 + 2y^2 + xyz,$$

find: (a) the force vector, (b) the force at the point $(1, -1, 2)$.

Solution:

Gradient:

$$\nabla U = (6x + yz)\hat{\imath} + (4y + xz)\hat{\jmath} + (xy)\hat{k}.$$

Force:

$$\vec{F} = -(6x + yz)\hat{\imath} - (4y + xz)\hat{\jmath} - (xy)\hat{k}.$$

At $(1, -1, 2)$:

$$\vec{F} = -(6 - 2)\hat{\imath} - (-4 + 2)\hat{\jmath} - (-1)\hat{k} = -4\hat{\imath} + 2\hat{\jmath} + \hat{k}.$$

Solved Problem 5: Equilibrium Analysis

Problem: For the potential

$$U = x^2 + y^2 - 2z^2,$$

determine whether the equilibrium at the origin is stable.

Solution:

Gradient:

$$\nabla U = 2x\hat{\imath} + 2y\hat{\jmath} - 4z\hat{k}.$$

Equilibrium occurs at $(0, 0, 0)$.

Since U increases in x, y directions but decreases in z, the equilibrium is a **saddle point** and therefore **unstable**.

Solved Problem 6: Work from Potential

Problem: Given

$$U = 5x^2 + 3y^2,$$

find the work done by the force when the particle moves from $(0, 0)$ to $(2, 1)$.

Solution:

Work by conservative force:

$$W = U_i - U_f.$$

Initial:

$$U_i = 0.$$

Final:

$$U_f = 5(2)^2 + 3(1)^2 = 23.$$

$$W = -23 \text{ J}.$$

3 Harmonic Oscillator in Two and Three Dimensions

The motion of a particle is specified by its position vector as a function of time. Velocity and acceleration are obtained by differentiating the position vector with respect to time.

The 3D harmonic oscillator potential is

$$U = \frac{1}{2}k(x^2 + y^2 + z^2).$$

Solved Problem 7: 3D Energy

Problem: A particle oscillates in three dimensions with amplitudes $A_x = 0.2$ m, $A_y = 0.1$ m, $A_z = 0.3$ m. If $k = 400$ N/m, find the total energy.

Solution:

$$E = \frac{1}{2}k(A_x^2 + A_y^2 + A_z^2) = \frac{1}{2}(400)(0.04 + 0.01 + 0.09) = 28 \text{ J}.$$

Solved Problem 8: Angular Frequency

Problem: A particle of mass 2 kg is subject to a 3D harmonic force with $k = 800$ N/m. Find the angular frequency.

Solution:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{800}{2}} = 20 \text{ rad/s}.$$

Solved Problem 9: Directional Independence

Problem: Explain mathematically why motion in different directions is independent.

Solution:

The equations of motion are:

$$m\ddot{x} + kx = 0, \quad m\ddot{y} + ky = 0, \quad m\ddot{z} + kz = 0.$$

Each equation depends on only one coordinate, so the motions are decoupled and independent.

Multiple Choice Questions (MCQs)

1. Motion in three dimensions is described using

   A) Scalars only

   B) Vectors only

   C) Both vectors and scalars

   D) Tensors only

   E) None of the above

2. The position vector of a particle in Cartesian coordinates is

   A) $\vec{r} = x + y + z$
   B) $\vec{r} = r\hat{r}$
   C) $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
   D) $\vec{r} = \sqrt{x^2 + y^2 + z^2}$
   E) None of the above

3. Velocity is defined as the time derivative of

   A) acceleration
   B) force
   C) position
   D) momentum
   E) energy

4. Acceleration is the time derivative of

   A) position
   B) velocity
   C) force
   D) displacement
   E) energy

5. Newton's second law in vector form is

   A) $F = ma$
   B) $\vec{F} = m\vec{a}$
   C) $\vec{F} = -\nabla U$
   D) $\vec{F} = -kx$
   E) None of the above

6. If the net force on a particle is zero, then

   A) its velocity must be zero
   B) its acceleration must be zero
   C) its momentum must be zero
   D) it must be at rest
   E) none of the above

7. Potential energy in three dimensions depends on

A) time
B) velocity
C) position
D) acceleration
E) force

8. Potential energy is a

A) vector
B) scalar
C) tensor
D) constant
E) none of the above

9. A conservative force can be expressed as

A) $\vec{F} = \nabla U$
B) $\vec{F} = -\nabla U$
C) $\vec{F} = U\nabla$
D) $\vec{F} = m\nabla$
E) None of the above

10. The del operator is written as

A) Δ
B) ∇
C) ∂
D) $\vec{d}$
E) None of the above

11. The gradient of a scalar field is a

A) scalar
B) vector
C) constant
D) tensor
E) none of the above

12. The gradient of potential energy points in the direction of

A) maximum decrease of U

B) maximum increase of U
C) zero change of U
D) constant force
E) none of the above

13. Equilibrium occurs at points where

A) U is maximum
B) U is minimum
C) $\nabla U = 0$
D) $\vec{F}$ is maximum
E) none of the above

14. The restoring force of a harmonic oscillator is

A) proportional to velocity
B) proportional to acceleration
C) proportional to displacement
D) constant
E) none of the above

15. Hooke's law is written as

A) $F = kx$
B) $F = -kx$
C) $F = ma$
D) $F = mg$
E) none of the above

16. The angular frequency of a harmonic oscillator depends on

A) amplitude
B) phase
C) mass and spring constant
D) displacement
E) time

17. The equation $m\ddot{x} + kx = 0$ describes

A) damped motion
B) forced motion

C) simple harmonic motion

D) uniform motion

E) none of the above

18. In two-dimensional harmonic motion, oscillations along different axes are

A) coupled

B) independent

C) random

D) circular only

E) none of the above

19. A Lissajous figure occurs when

A) motion is one-dimensional

B) frequencies are different

C) damping is large

D) force is zero

E) none of the above

20. In three dimensions, the harmonic oscillator force is

A) $\vec{F} = -k(x\hat{i} + y\hat{j} + z\hat{k})$

B) $\vec{F} = -kx$

C) $\vec{F} = m\vec{a}$

D) $\vec{F} = \nabla U$

E) none of the above

21. The potential energy of a 3D harmonic oscillator is

A) linear in position

B) quadratic in position

C) constant

D) zero

E) none of the above

22. An isotropic harmonic oscillator means

A) same mass in all directions

B) same frequency in all directions

C) zero damping

D) zero force

E) none of the above

23. The total energy of a multidimensional oscillator is

A) the product of energies

B) the sum of component energies

C) zero

D) time-dependent

E) none of the above

24. Energy is conserved in a harmonic oscillator when

A) damping exists

B) external force acts

C) no damping or driving force exists

D) resonance occurs

E) none of the above

25. The SI unit of spring constant is

A) N

B) J

C) kg

D) N/m

E) none of the above

26. Maximum potential energy occurs at

A) equilibrium

B) zero displacement

C) maximum displacement

D) maximum velocity

E) none of the above

27. Maximum kinetic energy occurs at

A) turning points

B) maximum displacement

C) equilibrium

D) zero velocity

E) none of the above

28. The del operator is widely used in

    A) mechanics only
    B) electromagnetism only
    C) electromagnetism and mechanics
    D) optics only
    E) none of the above

29. Motion in three dimensions generally requires

    A) one equation
    B) two equations
    C) three equations
    D) no equations
    E) none of the above