Mechanics — Lecture 4

Topic: Work, Energy, Conservative Forces, and Position-Dependent Forces

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1) Learning Outcomes

By the end of this lecture, you will be able to:

- Compute work for constant forces and forces that vary with position.
- Apply the work–energy theorem: $W_{\text{net}} = \Delta K$.
- Identify conservative vs. non-conservative forces and use energy methods efficiently.
- Use $F(x) = -\frac{dU}{dx}$ and interpret force/potential-energy graphs.
- Solve rich, multi-step problems (inclines, springs, friction, variable forces, turning points).

Expanded Conceptual Explanation and Applications

1. Work: Concept, Interpretation, and Applications

Work represents the mechanism by which energy is transferred to or from a system by the action of a force. In classical mechanics, work is defined as the dot (scalar) product of the force vector and the displacement vector.

$$W = \vec{F} \cdot \vec{d}$$

This vector definition emphasizes that only the component of the force parallel to the displacement contributes to the work done.

Using the definition of the dot product, the work can also be written in scalar form as

$$W = Fd\cos\theta$$

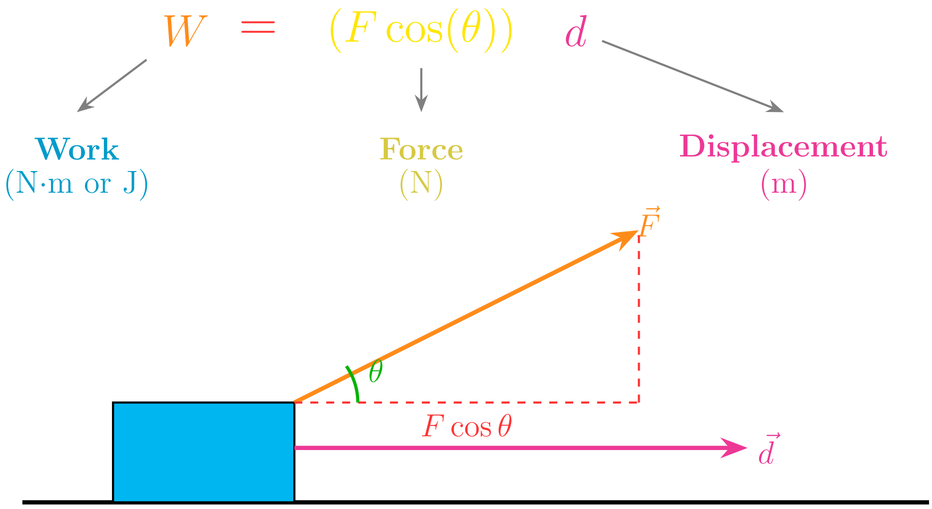
where F is the magnitude of the force, d is the magnitude of the displacement, and θ is the angle between the force and displacement vectors.

A crucial misconception among students is assuming that any applied force performs work. In reality, if the displacement is zero or if the force acts perpendicular to the displacement ($\theta = 90°$), the work done is zero.

This explains why centripetal forces, normal forces in circular motion, and static forces acting on an object at rest often do no work.

From a physical perspective, work modifies the kinetic energy of a system. If the work done is positive, the kinetic energy increases; if the work done is negative, the kinetic energy decreases.

Formula for Work



Problem 1: Motion under a Position-Dependent Force

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Solved Example: Variable force and stopping point

A particle of mass $m = 2\,\text{kg}$ moves along the x-axis under a force

$$F(x) = 10 - 2x \quad (\text{N})$$

where x is in meters. The particle starts from rest at $x = 0$.
(a) Find the speed of the particle at $x = 3\,\text{m}$. **(b)** Determine the position where the particle momentarily comes to rest.
Solution:
Work done from 0 to x:

$$W(x) = \int_0^x (10 - 2x)\,dx = \left[10x - x^2\right]_0^x = 10x - x^2$$

(a) At $x = 3$:

$$W(3) = 30 - 9 = 21\,\text{J}$$

Using work–energy theorem:

$$W = \Delta K = \frac{1}{2}mv^2 \Rightarrow 21 = \frac{1}{2}(2)v^2 \Rightarrow v = \sqrt{21} = 4.58\,\text{m/s}$$

(b) Turning point: At rest, $K = 0$, so total work must vanish:

$$10x - x^2 = 0 \Rightarrow x(10 - x) = 0$$

Non-trivial solution:

$$x = 10\,\text{m}$$

2. Work–Energy Theorem: Meaning and Use

The work–energy theorem provides a powerful connection between force-based and energy-based descriptions of motion. Rather than solving equations of motion, one can directly relate net work to changes in kinetic energy.

This theorem is particularly useful when forces vary, when motion occurs along curved paths, or when time is not easily determined.

Solved Example: Acceleration using work–energy

A $2\,\text{kg}$ object accelerates from 2 to $6\,\text{m/s}$.

$$W = \frac{1}{2}(2)(36 - 4) = 32\,\text{J}$$

Solved Example: Stopping by friction

A $5\,\text{kg}$ block with $v_0 = 4\,\text{m/s}$ stops due to friction.

$$K_0 = 40\,\text{J} \Rightarrow W_f = -40\,\text{J}$$

Solved Example: Speed after constant force

A 3 kg mass moves 5 m under 12 N.

$$W = 60 = \frac{1}{2}(3)v^2 \Rightarrow v = 6.32\,\text{m/s}$$

Solved Example: Motion under gravity

A 1 kg object falls 10 m.

$$W_g = mgh = 98\,\text{J}$$

Solved Example: Multiple forces

Forces of 8 N and −3 N act over 4 m.

$$W_{\text{net}} = 20\,\text{J}$$

Problem 2: Energy Method with Friction and Spring

Solved Example: Block–spring system with kinetic friction

A block of mass $m = 1.5\,\text{kg}$ slides on a rough horizontal surface with initial speed $v_0 = 6\,\text{m/s}$. The coefficient of kinetic friction is $\mu_k = 0.25$. The block compresses a spring of constant $k = 800\,\text{N/m}$ and momentarily comes to rest.
Find the maximum compression of the spring.
Solution:
Initial kinetic energy:

$$K_i = \frac{1}{2}mv_0^2 = \frac{1}{2}(1.5)(36) = 27\,\text{J}$$

Friction force:

$$f_k = \mu_k mg = 0.25(1.5)(9.8) = 3.675\,\text{N}$$

Let x be the maximum compression. Work done by friction:

$$W_f = -f_k x$$

Spring potential energy:

$$U_s = \frac{1}{2}kx^2$$

Energy equation:

$$K_i + W_f = U_s \Rightarrow 27 - 3.675x = \frac{1}{2}(800)x^2$$

$$400x^2 + 3.675x - 27 = 0$$

Solving quadratic:

$$x = \frac{-3.675 + \sqrt{(3.675)^2 + 4(400)(27)}}{800} \Rightarrow x = 0.26\,\text{m}$$

3. Conservative Forces and Potential Energy

A conservative force allows the definition of potential energy. This means energy can be stored and later recovered without loss. The defining characteristic of conservative forces is path independence.

Potential energy provides insight into stability, equilibrium, and allowed regions of motion.

Solved Example: Force from potential

If $U(x) = x^3$, then:

$$F = -3x^2$$

Solved Example: Potential from force

If $F(x) = -4x$, then:

$$U(x) = 2x^2$$

Solved Example: Equilibrium point

Given $U(x) = x^2$, equilibrium at:

$$\frac{\mathrm{d}U}{\mathrm{d}x} = 0 \Rightarrow x = 0$$

Solved Example: Stable vs unstable

For $U(x) = x^4 - x^2$, $x = 0$ unstable, $x = \pm 1/\sqrt{2}$ stable.

Solved Example: Turning points

With $E = 5$ and $U = x^2$, motion allowed for $|x| \leq \sqrt{5}$.

Problem 3: Motion in a Given Potential Energy Function

Solved Example: Analysis of motion from a potential-energy curve

A particle of mass 1 kg moves in one dimension with potential energy:

$$U(x) = x^4 - 4x^2 \quad \text{(J)}$$

(a) Find all equilibrium points. **(b)** Determine their stability. **(c)** If total energy $E = 3\,\text{J}$, find the allowed regions of motion.
Solution:
(a) Equilibrium:

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = -(4x^3 - 8x) = 0 \Rightarrow x(2 - x^2) = 0$$

$$x = 0,\ \pm\sqrt{2}$$

(b) Stability: Second derivative:

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = 12x^2 - 8$$

At $x = 0$: $\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = -8 < 0$ (unstable) At $x = \pm\sqrt{2}$: $\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = 16 > 0$ (stable)
(c) Allowed motion:

$$E \geq U(x) \Rightarrow 3 \geq x^4 - 4x^2$$

$$x^4 - 4x^2 - 3 \leq 0 \Rightarrow (x^2 - 3)(x^2 + 1) \leq 0$$

$$|x| \leq \sqrt{3}$$

4. Energy Conservation and Mechanical Energy

Energy conservation states that energy cannot be created or destroyed, only transformed. When non-conservative forces are absent, mechanical energy remains constant.

This principle greatly simplifies analysis of motion involving gravity and springs.

Solved Example: Free fall

$$v = \sqrt{2gh}$$

Solved Example: Spring compression

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

Solved Example: Incline without friction

$$mgh = \frac{1}{2}mv^2$$

Solved Example: With friction

$$mgh - W_f = \frac{1}{2}mv^2$$

Solved Example: Maximum height

$$h = \frac{v^2}{2g}$$

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Problem 4: Inclined Plane with Variable Force

Solved Example: Energy analysis with spatially varying force

A 2 kg block moves up a smooth incline under a force

$$F(x) = 12 - 3x \quad \text{(N)}$$

where x is measured along the incline. The block starts from rest at $x = 0$.
Find the maximum distance the block travels.
Solution:
Work done:

$$W = \int_0^x (12 - 3x)\,dx = \left[12x - \frac{3}{2}x^2\right]_0^x$$

At turning point $K = 0$, so $W = 0$:

$$12x - \frac{3}{2}x^2 = 0 \Rightarrow x(8 - x) = 0$$

$$x = 8\,\text{m}$$

Problem 5: Energy Loss and Recovery in Composite Motion

Solved Example: Block

A 3 kg block slides 4 m on a rough surface ($\mu_k = 0.2$), then enters a frictionless region and compresses a spring ($k = 600\,\text{N/m}$).
Find the maximum spring compression.
Solution:
Initial kinetic energy:

$$K_i = \frac{1}{2}(3)v^2$$

Energy lost to friction:

$$W_f = -\mu_k mgd = -0.2(3)(9.8)(4) = -23.5\,\text{J}$$

Remaining energy converts to spring energy:

$$\frac{1}{2}kx^2 = K_i - 23.5$$

$$x = \sqrt{\frac{2(K_i - 23.5)}{600}}$$

(Numerical value depends on initial speed.)

5) Multiple Choice Questions (30 MCQs)

Instruction: Choose the best answer. Option **E is always "None of them"**.

1. Work done by a constant force $\vec{F}$ over displacement $\vec{d}$ is:

- A) Fd
- B) $Fd\sin\theta$
- C) $\vec{F}\cdot\vec{d}$
- D) $Fd/\cos\theta$
- E) None of them

2. The SI unit of work is:

- A) Newton
- B) Pascal
- C) Watt
- D) Joule
- E) None of them

3. A force does zero work when it is:

- A) parallel to displacement
- B) opposite to displacement
- C) perpendicular to displacement
- D) larger than weight
- E) None of them

4. The work–energy theorem states that:

- A) work equals momentum change
- B) work equals force times velocity
- C) net work equals change in kinetic energy
- D) work equals potential energy
- E) None of them

5. Kinetic energy depends on:

- A) velocity only
- B) mass only
- C) acceleration
- D) mass and square of velocity

E) None of them

6. A conservative force is characterized by:

A) path-dependent work
B) time-dependent work
C) speed-dependent work
D) path-independent work
E) None of them

7. For a conservative force in one dimension:

A) $F = \frac{\mathrm{d}U}{\mathrm{d}x}$
B) $F = U/x$
C) $F = -\frac{\mathrm{d}U}{\mathrm{d}x}$
D) $F = \int U\,dx$
E) None of them

8. Near the Earth's surface, gravitational potential energy is:

A) $\frac{1}{2}mv^2$
B) $\frac{1}{2}kx^2$
C) mg/x
D) mgh
E) None of them

9. The spring force is given by Hooke's law:

A) $F = ma$
B) $F = mg$
C) $F = kx$
D) $F = -kx$
E) None of them

10. The potential energy stored in a spring is:

A) kx
B) k/x
C) mgx
D) $\frac{1}{2}kx^2$
E) None of them

11. Mechanical energy is defined as:

A) work plus force
B) kinetic energy only
C) potential energy only
D) sum of kinetic and potential energies
E) None of them

12. Mechanical energy is conserved when:

A) friction acts
B) non-conservative forces act
C) velocity is constant
D) only conservative forces act
E) None of them

13. A turning point in one-dimensional motion occurs when:

A) force is maximum
B) potential energy is minimum
C) velocity is maximum
D) kinetic energy is zero
E) None of them

14. The work done by kinetic friction is generally:

A) positive
B) zero
C) path-independent
D) negative
E) None of them

15. If net work on a particle is positive, its kinetic energy:

A) decreases
B) remains constant
C) becomes zero
D) increases
E) None of them

16. If a particle is momentarily at rest, its kinetic energy is:

A) maximum
B) negative
C) infinite
D) zero
E) None of them

17. The work done by gravity depends on:

A) the path taken
B) the speed of motion
C) the time taken
D) the vertical displacement
E) None of them

18. The force corresponding to a potential energy $U(x)$ is:

A) $\frac{\mathrm{d}U}{\mathrm{d}x}$
B) U/x
C) $-U$
D) $-\frac{\mathrm{d}U}{\mathrm{d}x}$
E) None of them

19. A particle can move only in regions where:

A) $U > E$
B) $E = 0$
C) force is zero
D) $E \geq U$
E) None of them

20. Stable equilibrium corresponds to:

A) maximum of U
B) zero of U
C) linear $U(x)$
D) minimum of U
E) None of them

21. The area under an F–x curve represents:

A) momentum
B) acceleration
C) potential energy
D) work
E) None of them

22. If velocity doubles, kinetic energy becomes:

A) double
B) triple
C) half
D) four times
E) None of them

23. A force that depends on position is written as:

A) $F = ma$
B) $F = mg$
C) $F = F(x)$
D) $F = at$
E) None of them

24. The work done by a centripetal force is:

A) positive
B) negative
C) path-dependent
D) zero
E) None of them

25. The SI unit of potential energy is:

A) Newton
B) Pascal
C) Watt
D) Joule
E) None of them

26. If $F(x) = 0$ for all x, the work done is:

A) infinite
B) negative

C) path-dependent

D) zero

E) None of them

27. Energy lost due to friction is converted mainly into:

A) kinetic energy

B) potential energy

C) mechanical energy

D) thermal energy

E) None of them

28. For a conservative force, work done over a closed path is:

A) maximum

B) minimum

C) non-zero

D) zero

E) None of them

29. Which of the following is a conservative force (ideal case)?

A) air resistance

B) kinetic friction

C) viscous drag

D) gravity

E) None of them