8 Solution of a System of Linear Equations

A system of linear equations is a set of equations where each equation is linear.

Consider the following system of m equations with n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

where a_{ij} are the coefficients, x_j are the variables, and b_i are the constants. In matrix form, the system can be expressed as:

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

There are several methods to solve this system:

- **Direct Methods:** Gauss Elimination Method
- **Iterative Methods:** Gauss-Seidel
- **Solution by Cramer's Rule**
- **Solution by Matrix Inversion**

8.1 Gauss Elimination Method

The Gauss Elimination Method is a systematic technique for solving systems of linear equations. It transforms the system's augmented matrix into an upper triangular form using a series of row operations, making it easier to solve through back-substitution.

Example 8.1. Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} 2x + 3y = 5 \\ 4x + y = 6 \end{cases}$$

Sol. Put the system in the following matrix form

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 1 & 6 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$$

$R_1 \Rightarrow \frac{R_1}{2}$

$$\left[\begin{array}{cc|c} \frac{2}{2} & \frac{3}{2} & \frac{5}{2} \\ 4 & 1 & 6 \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 4 & 1 & 6 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$$

$R_2 \Rightarrow R_2 - 4R_1$

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 4-4(1) & 1-4(\frac{3}{2}) & 6-4(\frac{5}{2}) \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -5 & -4 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$$

$R_2 \Rightarrow \frac{R_2}{-5}$

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & \frac{-5}{-5} & \frac{-4}{-5} \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & \frac{4}{5} \end{array}\right]$$

$$\Rightarrow y = \frac{4}{5} \Rightarrow x + \frac{3}{2}y = \frac{5}{2} \quad \Rightarrow x + \frac{3}{2}(\frac{4}{5}) = \frac{5}{2} \Rightarrow x + \frac{12}{10} = \frac{5}{2} \Rightarrow x = \frac{13}{10}$$

□

Example 8.2. Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} 3x - y + 2z = 12 \\ 3x + 2y + 3z = 11 \\ 2x - 2y - z = 2 \end{cases}$$

Sol. Put the system in the following matrix form

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 3 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_2 \Rightarrow R_2 - R_1$ and $R_3 \Rightarrow 3R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 3-3 & 2-(-1) & 3-2 & 11-12 \\ 3(2)-2(3) & -2(3)-2(-1) & -1(3)-2(2) & 2(3)-2(12) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 3 & 1 & -1 \\ 0 & -4 & -7 & -18 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \Rightarrow 3R_3 + 4R_1$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 3 & 1 & -1 \\ 0 & -4(3)+4(3) & -7(3)+4 & -18(3)-4 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -17 & -58 \end{array}\right]$$

$$\Rightarrow -17z = -58 \Rightarrow z = \frac{58}{17}$$

$$3y + z = -1 \Rightarrow 3y + \frac{58}{17} \Rightarrow 3y = -1 - \frac{58}{17} = -\frac{75}{17}$$

$$\Rightarrow y = -\frac{75}{17}(\frac{1}{3}) \Rightarrow y = -\frac{25}{17}$$

$$3x - y + 2z = 12 \Rightarrow 3x - (-\frac{25}{17}) + 2(\frac{58}{17}) = 12$$

$$3x + \frac{25}{17} + \frac{116}{17} = 12 \Rightarrow 3x = 12 - \frac{141}{17} = \frac{63}{17}$$

$$\Rightarrow x = \frac{63}{17}(\frac{1}{3}) = \frac{21}{17}$$

□

Homework of Gauss Elimination Method

1. Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} 3x - 3y = 2 \\ -7x + 2y = 0 \end{cases}$$

2. Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} x + 2y - 4z = 4 \\ 5x - 3y - 7z = 6 \\ 3x - 4y + 3z = 1 \end{cases}$$

8.2 Gauss Siedle Methods

The Gauss-Seidel Method is an iterative technique for solving systems of linear equations, typically used when the coefficient matrix is large and sparse. It uses the most recently updated values for each variable in each iteration, which can to faster convergence.

Consider a system of linear equations in matrix form:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Iteration Formula

The Gauss-Seidel Method updates each variable x_i as:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \dots, n$$

where $x_i^{(k+1)}$ is the updated value of x_i at the $(k+1)$-th iteration.

Consider the following system of linear equations:

$$\begin{cases} a_x x + a_y y + a_z z = d_1, \\ b_x x + b_y y + b_z z = d_2, \\ c_x x + c_y y + c_z z = d_3 \end{cases}$$

We can rewrite the equations in terms of x, y, and z for the Gauss-Seidel method:

$$x^{(k+1)} = \frac{1}{a_x} \left(d_1 - a_y y^{(k)} - a_z z^{(k)} \right),$$

$$y^{(k+1)} = \frac{1}{b_y} \left(d_2 - b_x x^{(k+1)} - b_z z^{(k)} \right),$$

$$z^{(k+1)} = \frac{1}{c_z} \left(d_3 - c_x x^{(k+1)} - c_y y^{(k+1)} \right).$$

Starting with initial guesses $x^{(0)} = 0$, $y^{(0)} = 0$, and $z^{(0)} = 0$, we apply the above formulas iteratively until the values converge.
where $a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z$ are constants.

Example 8.3. Solving systems of linear equations using Gauss Seidel method

$$\begin{cases} 5x - 2y + z = 4, \\ x + 4y - 2z = 3, \\ x + 4y + 4z = 17. \end{cases}$$

Sol.

$$x = \frac{1}{5}(4 + 2y - z)$$

$$y = \frac{1}{4}(3 - x + 2z)$$

$$z = \frac{1}{4}(17 - x - 4y)$$

$$x^{(k+1)} = \frac{1}{5}\left(4 + 2y^{(k)} - z^{(k)}\right),$$

$$y^{(k+1)} = \frac{1}{4}\left(3 - x^{(k+1)} + 2z^{(k)}\right),$$

$$z^{(k+1)} = \frac{1}{4}\left(17 - x^{(k+1)} - 4y^{(k+1)}\right).$$

We start with an initial guess: $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$.

Calculate $x^{(1)}$, $y^{(1)}$, $z^{(1)}$:

$$x^{(1)} = \frac{1}{5}(4 + 2 \times 0 - 0) = \frac{4}{5} = 0.8$$

$$y^{(1)} = \frac{1}{4}(3 - 0.8 + 2 \times 0) = \frac{2.2}{4} = 0.55$$

$$z^{(1)} = \frac{1}{4}(17 - 0.8 - 4 \times 0.55) = \frac{17 - 0.8 - 2.2}{4} = \frac{14}{4} = 3.5$$

After the first iteration, we have $x^{(1)} = 0.8$, $y^{(1)} = 0.55$, and $z^{(1)} = 3.5$.

Calculate $x^{(2)}$, $y^{(2)}$, $z^{(2)}$:

$$x^{(2)} = \frac{1}{5}(4 + 2 \times 0.55 - 3.5) = \frac{4 + 1.1 - 3.5}{5} = \frac{1.6}{5} = 0.32$$

$$y^{(2)} = \frac{1}{4}(3 - 0.32 + 2 \times 3.5) = \frac{3 - 0.32 + 7}{4} = \frac{9.68}{4} = 2.42$$

$$z^{(2)} = \frac{1}{4}(17 - 0.32 - 4 \times 2.42) = \frac{17 - 0.32 - 9.68}{4} = \frac{7}{4} = 1.75$$

□

Homework of Gauss Siedle Method

1. Solving systems of linear equations $(x^{(3)}, y^{(3)}, z^{(3)})$ using Gauss Seidel method

$$\begin{cases} 2x - y - 3z = 1, \\ 5x + 2y - 6z = 5, \\ 3x - y - 4z = 7. \end{cases}$$

2. Solving systems of linear equations $(x^{(3)}, y^{(3)}, z^{(3)})$ using Gauss Seidel method

$$\begin{cases} 2x - y + z = 1, \\ 3x - 2yz = 0, \\ 5x + y + 2z = 9. \end{cases}$$