7.3 Formation of PDE by Eliminating Arbitrary Constant

A PDE may formed by a eliminating arbitrary constants or arbitrary function from a given relation and other relation obtained by differentiating partially the given relation.

Remark 7.1. Suppose the following relation:

1. $\frac{\partial z}{\partial x} = z_x = p$
2. $\frac{\partial z}{\partial y} = z_y = q$

Example 7.4. Form a Partial Differential Equations from the following equation:

$$z = (x-a)^2 + (y-b)^2 \tag{1}$$

Sol.

$$z_x = 2(x-a) \Rightarrow (x-a) = \frac{z_x}{2} \Rightarrow -a = \frac{z_x}{2} - x \Rightarrow a = x - \frac{z_x}{2}$$
$$z_y = 2(y-b) \Rightarrow (y-b) = \frac{z_y}{2} \Rightarrow -b = \frac{z_y}{2} - y \Rightarrow b = y - \frac{z_y}{2}$$

Eq. (1) become

$$\begin{aligned} z &= (x - (x - \frac{z_x}{2}))^2 + (y - (y - \frac{z_y}{2}))^2 \\ &= (-\frac{z_x}{2})^2 + (-\frac{z_y}{2})^2 \\ &= \frac{z_x^2}{4} + \frac{z_y^2}{4} \Rightarrow 4z = z_x^2 + z_y^2 = p^2 + q^2 \end{aligned}$$

$$\therefore 4z = p^2 + q^2$$

□

Example 7.5. Form a Partial Differential Equations from the following equation:

$$z = f(x^2 + y^2) \tag{1}$$

Sol.

$$z_x = 2x \cdot f'(x^2 + y^2) \Rightarrow f'(x^2 + y^2) = \frac{z_x}{2x} \tag{2}$$

$$z_y = 2y \cdot f'(x^2 + y^2) \Rightarrow f'(x^2 + y^2) = \frac{z_y}{2y} \tag{3}$$

Sub. Eq. (2) in Eq (3)

$$z_y = 2y\frac{z_x}{2x} \Rightarrow \frac{z_y}{z_x} = \frac{y}{x}$$

$$\therefore \frac{q}{p} = \frac{y}{x}$$

□

Example 7.6. Form a Partial Differential Equations from the following equation:

$$z = ax + by + a^2 + b^2 \tag{1}$$

Sol.

$$z_x = a$$

$$z_y = b$$

Eq. (1) become

$$z = z_x x + z_y y + (z_x)^2 + (z_y)^2$$

$$\therefore z = px + qy + p^2 + q^2$$

□

Example 7.7. Form a Partial Differential Equations from the following equation:

$$v = f(x - ct) + g(x + ct) \tag{1}$$

Sol.

$$v_x = f'(x - ct) + g'(x + ct)$$

$$v_t = -cf'(x - ct) + cg'(x + ct)$$

$$v_{xx} = f''(x - ct) + g''(x + ct)$$

$$v_{tt} = c^2 f''(x - ct) + c^2 g''(x + ct)$$

$$v_{tt} = c^2(f''(x - ct) + g''(x + ct))$$

$$v_{tt} = c^2(v_{xx})$$

$$\therefore v_{tt} = c^2 v_{xx}$$

□

Homework of Formation of PDE by Eliminating Arbitrary Constant

Form a Partial Differential Equations from the following:

1. $z = ax + by + a^2 + b^2$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$
3. $z = f(\frac{y}{x})$
4. $f(x - at) + g(x + at)$

7.4 Method of Separation of Variables

Although there are several methods that can be tried to find particular solutions of a linear PDE, in the method of separation of variables we seek to find a particular solution of the form of a product of a function of x and a function of y.

$$u(x, y) = X(x)Y(y).$$

With this assumption, it is sometimes possible to reduce a linear PDE in two variables to two ODEs. To this end we observe that

$$\frac{\partial u}{\partial x} = X'Y \quad \frac{\partial u}{\partial y} = XY' \quad \frac{\partial^2 u}{\partial x^2} = X''Y \quad \frac{\partial^2 u}{\partial y^2} = XY'',$$

where the primes denote ordinary differentiation.

$$X' = \frac{dX}{dx} \quad Y' = \frac{dY}{dy}$$

Example 7.8. Solve the following Partial Differential Equation with boundary condition

$$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0 \tag{1}$$

With boundary condition

$$u(0, y) = 4e^{-2y} - 3e^{-6y} \tag{2}$$

To solve Eq. (1) suppose $u(x, y) = XY$.

Then

$$\frac{\partial u}{\partial x} = X'Y \quad \frac{\partial u}{\partial y} = XY'$$

$$X' = \frac{dX}{dx} \quad Y' = \frac{dY}{dy}$$

Put in Eq. (1)

$$YX' + 3XY' = 0$$
$$\frac{X'}{3X} = -\frac{Y'}{Y}$$

Now let

$$\frac{X'}{3X} = -\frac{Y'}{Y} = c \quad c \text{ constant}$$
$$\frac{X'}{3X} = c, \quad -\frac{Y'}{Y} = c$$

$$\Rightarrow X' = 3cX, \quad Y' = -cY, \Rightarrow X = a_1 e^{3cx}, \quad Y = a_2 e^{-cy},$$
$$\Rightarrow u(x,y) = XY = a_1 e^{3cx} a_2 e^{-cy} = a_1 a_2 e^{3cx-cy} = Be^{c(3x-y)}, \quad B = a_1 a_2$$

Now let

$$u(x,y) = u_1 + u_2 = b_1 e^{c_1(3x-y)} + b_2 e^{c_2(3x-y)}$$
$$\Rightarrow u(0,y) = b_1 e^{c_1(-y)} + b_2 e^{c_2(-y)} = 4e^{-2y} - 3e^{-6y}$$
$$\Rightarrow b_1 = 4, b_2 = -3, c_1 = 2, c_2 = 6$$
$$\therefore u(x,y) = 4e^{2(3x-y)} - 3e^{6(3x-y)}$$

Homework of Method of Separation of Variables

Form a Partial Differential Equations from the following:

1. Solve the following Partial Differential Equation with boundary condition

$$\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial y} = 0 \text{ With boundary condition } u(0,y) = 4e^{-6y} - 5e^{-y}$$

2. Solve the following Partial Differential Equation with boundary condition

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \text{ With boundary condition } u(0,y) = 2e^{-6y} + e^{-2y}$$

7.4.1 Heat Equation

The heat equation is a fundamental partial differential equation that describes how heat (or temperature) evolves over time in a given region. It is widely used in physics, engineering, and mathematics to model heat conduction.

The One-Dimensional Heat Equation

In its simplest form, the one-dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x,t)$ is the temperature distribution function, representing the temperature at position x and time t.
- k is the thermal diffusivity constant, which depends on the material properties.
- $\frac{\partial u}{\partial t}$ represents the rate of change of temperature with respect to time.
- $\frac{\partial^2 u}{\partial x^2}$ represents the spatial second derivative of temperature, indicating how temperature varies along the spatial axis.

Solution of the Heat Equation

The method of separation of variables is commonly used to solve the heat equation, leading to solutions of the form:

$$u(x,t) = X(x)T(t)$$

Substituting $u(x,t) = X(x)T(t)$ into the heat equation and separating the variables results in two ordinary differential equations, one for $X(x)$ and one for $T(t)$, which can be solved individually under the given initial and boundary conditions.

Example 7.9 (Heat Equation)**.** Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

With boundary condition

$$u(0,t) = 0, u(10,t) = 0 \tag{2}$$

Sol. To solve Eq. (1) suppose $u(x,t) = XT$.

Then

$$\frac{\partial u}{\partial t} = XT' \quad \frac{\partial^2 u}{\partial X^2} = X''T$$

Since $u(0,t) = 0, u(10,t) = 0 \Rightarrow X(0) = 0, X(10) = 0$

Put in Eq. (1)

$$XT' = 2X''T$$
$$\frac{T'}{2T} = \frac{X''}{X}$$

Now let

$$\frac{T'}{2T} = \frac{X''}{X} = b, \quad b \text{ constant}$$
$$\frac{T'}{2T} = b \Rightarrow T' = 2bT$$
$$\frac{X''}{X} = b \Rightarrow X'' = 2bX \Rightarrow X'' - 2bX = 0$$

Now let $b = \lambda^2$

when $\lambda^2 \geq 0$ trivial solution. Then $b = -\lambda^2 < 0$

$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

Since $X(0) = 0$,

$$\Rightarrow X(0) = A\cos(\lambda(0)) + B\sin(\lambda(0)) = A \Rightarrow A = 0,$$

$$X(10) = A\cos(\lambda(10)) + B\sin(\lambda(10)) = 0 \Rightarrow X(10) = B\sin(\lambda(10))$$

$$\therefore \sin(\lambda(10)) = 0$$

Since $\sin(\lambda(10)) = 0 \Rightarrow 10\lambda = n\pi, n = 0, 1, 2, \ldots$

$$\Rightarrow \lambda = \frac{n\pi}{10}$$

$$\therefore x(x) = B_n \sin(\frac{n\pi}{10}x)$$

Since $T' = 2bT \Rightarrow T' = e^{2bt} \Rightarrow T' = e^{-2\lambda^2 t} \Rightarrow T' = e^{-2(\frac{n\pi}{10})^2 t}$

$$\Rightarrow T = C_n e^{-2(\frac{n\pi}{10})^2 t}$$

$$\therefore u(x,t) = XT = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{10}x) e^{-2(\frac{n\pi}{10})^2 t}, A_n = B_n C_n$$

□

Homework of Heat Equation

1. Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = 0, \quad k \text{ constant}$$

   With boundary condition

$$u(0,t) = 0, u(L,t) = 0$$

2. Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = 6\frac{\partial^2 u}{\partial x^2} = 0 \text{ With boundary condition } u(0,t) = 0, u(30,t) = 0$$

7.4.2 Wave Equation

The wave equation is a second-order partial differential equation that describes the propagation of waves, such as sound waves, light waves, or water waves, in a given medium. It is a fundamental equation in physics and engineering, with applications in fields such as acoustics, electromagnetism, and fluid dynamics.

The Wave Equation in One Dimension

In one-dimensional space, the wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x, t)$ represents the displacement of the wave at position x and time t,
- c is the speed of wave propagation in the medium.
- Second Partial Derivative with Respect to Time ($\frac{\partial^2 u}{\partial t^2}$):
  - This term represents the **acceleration** of the wave function $u(x, t)$ with respect to time at any given position x.
  - It indicates how the wave's displacement changes over time, capturing the oscillatory nature of waves.
- Second Partial Derivative with Respect to Space ($\frac{\partial^2 u}{\partial x^2}$):
  - This term measures the **curvature** of the wave function $u(x, t)$ with respect to spatial dimensions.
  - Physically, it represents how the wave's displacement changes along the spatial dimension(s), indicating how the wave bends or curves at any given point.

Solution of the Wave Equation

The method of separation of variables is commonly used to solve the wave equation, leading to solutions of the form:

$$u(x,t) = X(x)T(t)$$

Substituting $u(x,t) = X(x)T(t)$ into the heat equation and separating the variables results in two ordinary differential equations, one for $X(x)$ and one for $T(t)$, which can be solved individually under the given initial and boundary conditions.

Example 7.10. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

With boundary condition

$$u(0,t) = 0, u(4,t) = 0, u(x,0) = 5\sin(\frac{\pi}{4}x), u_t(x,0) = 0 \tag{2}$$

Sol. To solve Eq. (1) suppose $u(x,t) = XT$.

$$\frac{\partial^2 u}{\partial t^2} = XT'' \quad \frac{\partial^2 u}{\partial X^2} = X''T$$

Put in Eq. (1)

$$\begin{aligned}
&\text{Since } u(0,4),\\
&\Rightarrow u(4,t) = C_2\sin(4p)(C_3\cos(pt) + C_4\sin(pt)) = 0\\
&\text{Since } C_2 \neq 0 \Rightarrow \sin(4p) = 0 \Rightarrow 4p = n\pi \Rightarrow p = \frac{n\pi}{4}, n = 0, 1, \cdots\\
&\Rightarrow u(x,t) = C_2\sin(\frac{n\pi}{4}x)(C_3\cos(\frac{n\pi}{4}t) + C_4\sin(\frac{n\pi}{4}t))\\
u_t(x,t) &= C_2\sin(\frac{n\pi}{4}x)(-\frac{n\pi}{4}C_3\sin(\frac{n\pi}{4}t) + \frac{n\pi}{4}C_4\cos(\frac{n\pi}{4}t))
\end{aligned}$$

Since $u_t(x,0) = 0$

$$\Rightarrow u_t(x,t) = C_2 \sin(\frac{n\pi}{4}x)(\frac{n\pi}{4}C_4)$$

Since $C_2 \neq 0 \Rightarrow C_4 = 0$

$$\Rightarrow u(x,t) = C_2 \sin(\frac{n\pi}{4}x)(C_3 \cos(\frac{n\pi}{4}t) = C_2C_3 \sin(\frac{n\pi}{4}x)\cos(\frac{n\pi}{4}t)$$

Since $u(x,0) = 5\sin(\frac{\pi}{4}x)$

$$\Rightarrow u(x,0) = C_2C_3 \sin(\frac{n\pi}{4}x) = 5\sin(\frac{\pi}{4}x) \Rightarrow C_1C_2 = 5$$

$$\therefore u(x,t) = 5\sin(\frac{n\pi}{4}x)\cos(\frac{n\pi}{4}t)$$

□

Homework of Wave Equation

1. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2} = 0$$

With boundary condition

$$u(0,t) = 0, u(k^2,t) = 0, u(x,0) = \sin(\frac{\pi}{k^2}x), u_t(x,0) = 0$$

2. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} = 0$$

With boundary condition

$$u(0,t) = 0, u(9,t) = 0, u(x,0) = 2\sin(\frac{\pi}{9}x), u_t(x,0) = 0$$