2 Algebra Preliminaries

2.1 Sets

Definition 2.1. A **set** is a well-defined collection of distinct objects, called **elements**, enclosed in curly brackets $\{\}$.

Formal Definition: A set S is defined as $S = \{a, b, c, \dots\}$, where each element is unique and well-defined.

Examples of Sets

- **Finite Set:** $A = \{1, 2, 3, 4, 5\}$
- **Infinite Set:** $B = \{1, 2, 3, \dots\}$
- **Empty Set (Null Set):** $\emptyset = \{\}$ (A set with no elements)

Common Sets:

$\mathbb{N}$ - Natural numbers, $\mathbb{Z}$ - Integers, $\mathbb{Q}$ - Rational numbers, $\mathbb{R}$ - Real numbers, $\mathbb{C}$ - Complex numbers.

Relations and Membership:

- $x \in A$ (Element of A), $y \notin B$ (Not an element of B)
- $A \subseteq B$ (A is a Subset of B), $A \subset B$ (A is a Proper Subset of B), $A = B$ (Equality)

2.2 Integer and Natural Numbers

The set $\mathbb{Z}$ of all integers, consists of all positive and negative integers as well as 0. Thus $\mathbb{Z}$ is the set given by

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

While the set of all positive integers (Natural Numbers), denoted by $\mathbb{N}$, is defined by

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

2.2.1 Basic Properties of Natural Numbers

Addition in $\mathbb{N}$

- **Closure:** For any $a, b \in \mathbb{N}$, the sum $a + b$ is also in $\mathbb{N}$.
- **Associativity:** $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{N}$.
- **Commutativity:** $a + b = b + a$ for all $a, b \in \mathbb{N}$.
- **Cancellation Law:** For any $a, b, c \in \mathbb{N}$, if $a + c = b + c$, then $a = b$.

Multiplication in $\mathbb{N}$

- **Closure:** For any $a, b \in \mathbb{N}$, the product $a \cdot b$ is also in $\mathbb{N}$.
- **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{N}$.
- **Commutativity:** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{N}$.
- **Identity Element:** 1 serves as the multiplicative identity since $a \cdot 1 = a$ for every $a \in \mathbb{N}$.
- **Cancellation Law:** For any $a, b, c \in \mathbb{N}$, if $a \cdot c = b \cdot c$, then $a = b$.
- **Distributivity:** Multiplication is distributive over addition: For any $a, b, c \in \mathbb{N}$,

$$a(b + c) = ab + ac.$$

Subtraction and Division in $\mathbb{N}$

- **Subtraction:** The operation of subtraction is *not* always closed in $\mathbb{N}$. For example, $2 - 5$ is not a natural number.
- **Division:** Similarly, division is not generally closed in $\mathbb{N}$; for instance, $3 \div 2$ does not yield a natural number.

2.2.2 Basic Properties of Integer Numbers

Addition in $\mathbb{Z}$

- **Closure:** For any $a, b \in \mathbb{Z}$, the sum $a + b$ is also in $\mathbb{Z}$.
- **Associativity:** $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Z}$.
- **Commutativity:** $a + b = b + a$ for all $a, b \in \mathbb{Z}$.
- **Identity Element:** 0 is the additive identity since $a + 0 = a$ for every $a \in \mathbb{Z}$.
- **Inverses:** Every integer a has an inverse $-a$ such that $a + (-a) = 0$.

Subtraction in $\mathbb{Z}$

Subtraction is always defined in $\mathbb{Z}$ because for any $a, b \in \mathbb{Z}$, the difference $a - b = a + (-b)$ is also an integer.

Multiplication in $\mathbb{Z}$

- **Closure:** For any $a, b \in \mathbb{Z}$, the product $a \cdot b$ is in $\mathbb{Z}$.
- **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{Z}$.
- **Commutativity:** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Z}$.
- **Identity Element:** 1 is the multiplicative identity since $a \cdot 1 = a$ for every $a \in \mathbb{Z}$.
- **Cancellation Law:** For any $a, b, c \in \mathbb{Z}$, if $a \cdot c = b \cdot c$, then $a = b$.
- **Distributivity:** Multiplication is distributive over addition: For any $a, b, c \in \mathbb{Z}$,

$$a(b + c) = ab + ac.$$

Division in $\mathbb{Z}$

Division is not a closed operation in $\mathbb{Z}$. For example, $3 \div 2$ is not an integer.

Important Theorem

Theorem 2.1. *Let $a, b \in Z$, Then:*

1. $a \cdot 0 = 0 \cdot a = 0$
2. $(-a)b = a(-b) = -ab$
3. $(-a)(-b) = ab$

Proof. 1. $0 + 0 = 0$ (Identity element in $\mathbb{Z}$)

$\Rightarrow (0 + 0)a = 0a \Rightarrow 0a + 0a = 0a$

$\Rightarrow 0a + 0a + (-0a) = 0a + (-0a)$ (inverse in $\mathbb{Z}$)

$\Rightarrow 0a = 0$

Similarly $a0 = 0$

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2. $b + (-b) = 0$ (inverse in $\mathbb{Z}$)

$\Rightarrow a(b + (-b)) = a0 = 0$ (From (1))

$\Rightarrow ab + a(-b) = ab + (-ab) \Rightarrow a(-b) = -ab$

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3. $(-a)(-b) = ab$

In (2), replace a by $(-a) \Rightarrow (-a)(-b) = -((-a)b) = -(-ab) = ab$

□

2.2.3 Laws of Exponents

For $n, m \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, we have the following exponentiation rules:

1. **Product Rule:** $a^m \cdot a^n = a^{m+n}$
2. **Quotient Rule:** $\frac{a^m}{a^n} = a^{m-n}$, for $m \geq n,\ a \neq 0$
3. **Power of a Power:** $(a^m)^n = a^{m \cdot n}$
4. **Power of a Product:** $(ab)^n = a^n \cdot b^n$

2.2.4 Properties of Inequalities

For $a, b, c \in \mathbb{Z}$, the following properties hold:

1. **Transitivity:** If $a < b$ and $b < c$, then $a < c$.
2. **Addition Property:** If $a < b$, then $a + c < b + c$ for any $c \in \mathbb{Z}$.
3. **Multiplication by a Positive Number:** If $a < b$ and $c > 0$, then $ac < bc$.
4. **Multiplication by a Negative Number:** If $a < b$ and $c < 0$, then $ac > bc$ (the inequality sign reverses).

2.3 Even and Odd Numbers

Even Numbers

An integer n is called *even* if it is divisible by 2. That is, n is even if there exists an integer k such that:

$$n = 2k.$$

Examples: $2 = 2(1)$, $4 = 2(2)$, $10 = 2(5)$.

Odd Numbers

An integer n is called *odd* if it is not divisible by 2. Formally, n is odd if it can be expressed as:

$$n = 2k + 1,$$

where k is an integer.

Examples:

- $1 = 2(0) + 1$
- $3 = 2(1) + 1$
- $7 = 2(3) + 1$