7 Partial Differential Equations

Partial Differential Equation (PDE)is an equation that involves multiple independent variables, an unknown function that depends on these variables, and partial derivatives of the unknown function. PDEs are used to formulate problems involving functions of several variables and are especially important in describing physical phenomena such as heat conduction, wave propagation, fluid flow, and electromagnetism.

A general form of a PDE can be written as:

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1^2}, \ldots\right) = 0$$

where $u = u(x_1, x_2, \ldots, x_n)$ is the unknown function, and F represents a relationship between u and its partial derivatives.

7.1 Classification of Partial Differential Equations

Partial differential equations are classified according to many things. Classification is an important concept because the general theory and methods of solution usually apply only to a given class of equations.

7.1.1 Order of the Partial Differential Equations

The order of a PDE is **the order of the highest partial derivative** in the equation.

For example:

1. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \; or \; u_t + cu_x = 0$ **(First-order)**
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; or \; u_{xx} + u_{yy} = 0$ **(Second-order)**
3. $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$ **(First-order)**
4. $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 0 \; or \; u_{xxx} + u_{xy} + u_y = 0$ **(Third-order)**

7.1.2 Degree of Partial Differential Equations

The degree of a partial differential equation is **the degree of the Highest order partial derivative** occurring in the equation.

For example:

1. $u_{xx} + u_y^2 + u = 0$ **(First-degree)**
2. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ **(First-degree)**
3. $u_t + uu_x = 0$ **(Second-degree)**
4. $\left(\frac{\partial^4 u}{\partial x^4}\right)^3 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial u}{\partial x} = 0$ **(Third-degree)**
5. $e^{\frac{\partial u}{\partial x}} + \sin\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$ **(Undefined)**

Homework: Determine the Order and Degree of the PDEs

For each of the following PDEs, determine the order and degree:

1. $\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} = 0$
2. $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 1$
3. $\frac{\partial^3 u}{\partial t^3} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$
4. $\frac{\partial^3 u}{\partial x^3} + u^2\frac{\partial u}{\partial t} = \sin(x)$
5. $u_{xx} + 2u_{yy} + 3u_{xy} + 4 = 0$

7.1.3 Number of Variables of Partial Differential Equations

1. $u_t = u_x$ **(Two variables: x and t)**
2. $u_t = u_{tt} + \frac{1}{r}u_r + u_\theta$ **(Three variables: r, θ, and t)**

Homework: Identify the Number of Variables

For each of the following PDEs, identify the number of independent variables:

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$
2. $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \alpha u$
3. $\frac{\partial u}{\partial t} + \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^2 u}{\partial y^2} = f(x, y, t)$

7.1.4 Linearity Partial Differential Equations

A PDE is linear if the unknown function and its partial derivatives appear to the first power and are not multiplied together. More precisely, a second order linear equation in two variables is an equation of the form:

$$a(x, y, z, \dots) * u_{xx} + b(x, y, z, \dots) * u_{xy} + \dots + f(x, y, z, \dots) * u = g(x, y, z, \dots)$$

where u is the unknown function, $x, y, z, \dots$ are the independent variables, and $a, b, c, \dots$ and f, g are functions of the independent variables only.

For example:

1. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ **(Linear)**
2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **(Linear)**
3. $u_{tt} = e^{-t} u_{xx} = sin(x)$ **(Linear)**
4. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ **(Non-Linear)**
5. $u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 0$ **(Non-Linear)**

7.1.5 Homogeneity of Partial Differential Equations

A PDE is classified as either **Homogeneous** or **Non-Homogeneous** based on the presence of a non-zero term that does not involve the unknown function or its derivatives. A PDE is said to be **Homogeneous** if all terms involve either the unknown function or its derivatives, and there is no standalone term (i.e., the right-hand side of the equation is zero).

In contrast, a PDE is called **Non-Homogeneous** if there is a non-zero term on the right-hand side of the equation that is independent of the unknown function.

For example:

1. $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$ **(Homogeneous)**
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **(Homogeneous)**
3. $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = g(x,t)$ **(Non-homogeneous)**
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ **(Non-homogeneous)**

Homework: Linearity and Homogeneity

Classify the following PDEs as linear or nonlinear, and as homogeneous or non-homogeneous:

1. $\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$
2. $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$
3. $\frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial y^2} + g(x,y) = 0$
4. $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta u = 0$

7.2 Solution of Partial Differential Equations

Partial Differential Equations (PDEs) arise in various fields such as physics, engineering, and finance. The solutions to these equations can describe physical phenomena like heat conduction, wave propagation, and fluid dynamics. In the section presents several methods to solve PDEs along with examples.

7.2.1 Solution of Partial Differential Equations by Direct Integration

Direct integration is a technique used to solve certain types of first-degree Partial Differential Equations (PDEs) where the equation can be rearranged to allow for straightforward integration with respect to one variable.

Example 7.1. Find the solution of the following PDE

$$\frac{\partial^2 z}{\partial x^2} = xy$$

Sol.

$$\frac{\partial^2 z}{\partial x^2} = xy \tag{1}$$

Integrate Equation (1) w.r.t. x.

$$\frac{\partial z}{\partial x} = \int xy\, dx + f(y) = (\frac{x^2}{2})y + f(y)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \tag{2}$$

Now, Integrate Equation (2) w.r.t. x.

$$\begin{aligned} z &= \int \frac{x^2y}{2} + f(y)\,dx + g(y) \\ &= \frac{y}{2}\int x^2 + f(y)\,dx + g(y) \\ &= \frac{y}{2}\int x^2\,dx + \int f(y)\,dx + g(y) \\ &= \frac{y}{2}(\frac{x^3}{3}) + x \cdot f(y) + g(y) \end{aligned}$$

$$\therefore z = \frac{x^3y}{6} + x \cdot f(y) + g(y)$$

Where $f(y), g(y)$ is arbitrary parametric. □

Example 7.2. Find the solution of the following PDE

$$\frac{\partial^2 z}{\partial xy} = 0$$

Sol.

$$\frac{\partial^2 z}{\partial xy} = 0 \tag{1}$$

Integrate Equation (1) w.r.t. x.

$$\frac{\partial z}{\partial y} = \int 0\,dx + f(y) = f(y)$$

$$\therefore \frac{\partial z}{\partial y} = f(y) \tag{2}$$

Now, Integrate Equation (2) w.r.t. y.

$$z = \int f(y)\,dy + g(x)$$

Where $f(y), g(y)$ is arbitrary parametric. □

Example 7.3. Find the solution of the following PDE

$$\frac{\partial^2 u}{\partial xy} = 6x + 12y^2$$

With boundary condition, $u(1, y) = y^2 - 2y$, $u(x, 2) = 5x - 5$

Sol.

$$\frac{\partial^2 u}{\partial xy} = 6x + 12y^2 \tag{1}$$

Integrate Equation (1) w.r.t. x.

$$\frac{\partial u}{\partial y} = \int 6x + 12y^2\, dx + f(y) = \frac{6x^2}{2} + 12y^2 x + f(y)$$

$$\therefore \frac{\partial u}{\partial y} = 3x^2 + 12y^2 x + f(y) \tag{2}$$

Now, Integrate Equation (2) w.r.t. y.

$$\begin{aligned} u &= \int 3x^2 + 12y^2 x + f(y)\, dy + g(x) \\ &= 3x^2 y + \frac{12y^3}{3} x + \int f(y)\, dy + g(x) \\ &= 3x^2 y + 4y^3 x + h(y) + g(x) \quad \text{where } h(y) = \int f(y)\, dy \end{aligned}$$

$$\begin{aligned} u(x, y) &= 3x^2 y + 4y^3 x + h(y) + g(x) \\ u(1, y) &= 3(1)^2 y + 4y^3(1) + h(y) + g(1) = y^2 - 2y \\ &= 3y + 4y^3 + h(y) + g(1) = y^2 - 2y \\ h(y) &= -4y^3 + y^2 - 5y - g(1) \end{aligned}$$

$$\therefore u(x, y) = 3x^2 y + 4y^3 x + (-4y^3 + y^2 - 5y - g(1)) + g(x)$$

$$u(x,y) = 3x^2y + 4y^3x - 4y^3 + y^2 - 5y - g(1) + g(x)$$

$$u(x,2) = 3x^2(2) + 4(2)^3x - 4(2)^3 + (2)^2 - 5(2) - g(1) + g(x) = 5x - 5$$

$$= 6x^2 + 32x - 32 + 4 - 10 - g(1) + g(x) = 5x - 5$$

$$= 6x^2 + 32x - 38 - g(1) + g(x) = 5x - 5$$

$$g(x) = -6x^2 - 27x + 33 + g(1)$$

$$u(x,y) = 3x^2y + 4y^3x - 4y^3 + y^2 - 5y - g(1) + (-6x^2 - 27x + 33 + g(1))$$

$$\therefore u(x,y) = 3x^2y + 4y^3x - 4y^3 + y^2 - 5y - 6x^2 - 27x + 33$$

□

Homework: Solving Partial Differential Equations by Direct Integration

Solve the following PDEs by direct integration:

1. $\frac{\partial u}{\partial x} = y^2$
2. $\frac{\partial u}{\partial t} = 3x^2$, with the boundary condition $u(x,0) = x^3$.
3. $\frac{\partial u}{\partial t} = -4x$, with the boundary condition $u(x,0) = e^x$.