5 Numerical Differentiation

Numerical differentiation is a method used to approximate the derivative of a function when an analytical solution is difficult or impossible to obtain. It's an important technique in numerical analysis and computational mathematics. Here's an overview of numerical differentiation:

1. **Basic Concept:** The derivative of a function $f(x)$ at a point x is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   Numerical differentiation approximates this limit using finite differences.

2. **Finite Differences:** Finite differences are a numerical method used to approximate derivatives and solve difference equations. Here are some common forms of finite differences and their definitions

   - **Forward Difference:** $f'(x) \approx \frac{f(x+h)-f(x)}{h}$. This method uses the function value at x and $(x+h)$.
   - **Backward Difference:** $f'(x) \approx \frac{f(x)-f(x-h)}{h}$. This method uses the function value at x and $(x-h)$.
   - **Central Difference:** $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$. This method is generally more accurate as it uses the function values at both $(x+h)$ and $(x-h)$.

3. **Higher-Order Derivatives:** For more accuracy, higher-order methods can be used. These methods involve more points and higher-order terms in the Taylor series expansion.

6 Solutions of Ordinary Differential Equation

Numerical differentiation is often used in the context of solving ordinary differential equations (ODEs) when analytical solutions are difficult or impossible to obtain.

Solving ODEs numerically involves finding the function $y(t)$ that satisfies the differential equation $\frac{dy}{dt} = f(t, y)$ given an initial condition $y(t_0) = y_0$. Here are some common numerical methods for solving ODEs:

1. Euler Method
2. Modified Euler Method
3. Rung Kutta Method
4. Rung Kutta-Merson Method

6.1 Euler Method

The Euler method is a numerical technique for solving ordinary differential equations (ODEs) with a given initial value. It is defined as follows:

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The Euler method updates the solution using the formula:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where h is the step size, t_n is the current time, and y_n is the current value of the solution.

The Step by Step Method

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

Step 1. We find t_i

$$t_1 = t_0 + h$$
$$t_2 = t_1 + h$$
$$\vdots$$
$$t_{n+1} = t_n + h$$

Step 2. We find y_i

$$y_1 = y_0 + h \cdot f(t_0, y_0)$$
$$y_2 = y_1 + h \cdot f(t_1, y_1)$$
$$\vdots$$
$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

Example 6.1. Use Euler's method to solve the $D.E.$ and find y_3

$$\frac{dy}{dt} = \frac{t^2 - y}{2}, \quad \text{with } t_0 = 0, y_0 = 4, h = 0.05$$

Sol. $f(t, y) = \frac{t^2 - y}{2}$

Step 1. We find t_i

$$t_0 = 0$$
$$t_{n+1} = t_n + h$$
$$t_1 = t_0 + h = 0 + 0.05 = 0.05$$
$$t_2 = t_1 + h = 0.05 + 0.05 = 0.1$$

Step 2. We find y_i

$$y_0 = 4$$

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

$$f(t_0, y_0) = \frac{t_0^2 - y_0}{2} = \frac{0^2 - 4}{2} = -2$$

$$y_1 = y_0 + h \cdot f(t_0, y_0) = 4 + 0.05(-2) = 4 + (-0.1) = 3.9$$

$$f(t_1, y_1) = \frac{t_1^2 - y_1}{2} = \frac{(0.05)^2 - 3.9}{2} = -1.9488$$

$$y_2 = y_1 + h \cdot f(t_1, y_1) = 3.9 + 0.05(-1.9488) = 3.9 + (-0.0974) = 3.8026$$

$$f(t_2, y_2) = \frac{t_2^2 - y_2}{2} = \frac{(0.1)^2 - 3.8026}{2} = -1.8963$$

$$y_3 = y_2 + h \cdot f(t_1, y_1) = 3.8026 + 0.05(-1.8963) = 3.8026 + (-0.0948) = 3.7077$$

□

Example 6.2. Use Euler's method to solve the $D.E.$ and find y_4

$$\frac{dy}{dt} + 2y = 1.3e^{-t}, \quad \text{with } t_0 = 0, y_0 = 0.5, h = 1$$

Sol. $\frac{dy}{dt} + 2y = 1.3e^{-t} \Rightarrow \frac{dy}{dt} = 1.3e^{-t} - 2y \Rightarrow f(t, y) = 1.3e^{-t} - 2y$

Step 1. We find t_i

$$t_0 = 0$$

$$t_{n+1} = t_n + h$$

$$t_1 = t_0 + h = 0 + 1 = 1$$

$$t_2 = t_1 + h = 1 + 1 = 2$$

$$t_3 = t_3 + h = 2 + 1 = 3$$

Step 2. We find y_i

$$y_0 = 0.5$$

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

$$f(t_0, y_0) = 1.3e^{-t_0} - 2y_0 = 1.3e^{-0} - 2(0.5) = 0.3$$

$$y_1 = y_0 + h \cdot f(t_0, y_0) = 0.5 + (1)(0.3) = 0.5 + 0.3 = 0.8$$

$$f(t_1, y_1) = 1.3e^{-t_1} - 2y_1 = 1.3e^{-1} - 2(0.8) = -1.1218$$

$$y_2 = y_1 + h \cdot f(t_0, y_0) = 0.8 + (-1.1218) = -0.3218$$

$$f(t_2, y_2) = 1.3e^{-t_2} - 2y_2 = 1.3e^{-2} - 2(-0.3218) = 0.8194$$

$$y_3 = y_2 + h \cdot f(t_2, y_2) = -0.3218 + 0.8194 = 0.4977$$

$$f(t_3, y_3) = 1.3e^{-t_3} - 2y_3 = 1.3e^{-3} - 2(0.4977) = -0.9307$$

$$y_4 = y_3 + h \cdot f(t_2, y_2) = 0.4977 + (-0.9307) = -0.433$$

□

Homework of Euler Method

1. Use Euler's method to solve $\frac{dy}{dt} = t - y^2$ and find y_5, when $t_0 = 0, y_0 = 1, h = 0.1$.

2. Use Euler's method to solve the $D.E.$ and find y_4

$$\frac{dy}{dx} + y = -xy^2, \quad \text{with } x_0 = 0, y_0 = 1, h = 0.1$$

6.2 Modified Euler Method

The Modified Euler Method gives from modified the value of $(y_n + 1)$ at point $(x_n + 1)$ by gives the new value $(y_n + 1)$.

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The method involves the following steps:

1. **Predictor Step (Euler Method)**:

$$y^*_{n+1} = y_n + hf(t_n, y_n)$$

2. **Corrector Step (Modified Euler's Method)**:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right]$$

The Step by Step Method

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

Step 1. Using Euler's method $y^*_{n+1} = y_n + hf(t_n, y_n)$

Step 2. Using $y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right]$

Example 6.3. Use Modified Euler's method to solve the $D.E.$ and find y_2

$$\frac{dy}{dt} = \frac{t^2 - y}{2}, \quad \text{with } t_0 = 0, y_0 = 4, h = 0.05$$

Sol. $f(t,y) = \frac{t^2-y}{2}$

Step 1. We find t_i and y_i^*

$$t_0 = 0 \qquad t_{n+1} = t_n + h$$

$$t_1 = t_0 + h = 0 + 0.05 = 0.05 \qquad t_2 = t_1 + h = 0.05 + 0.05 = 0.1$$

$$y_0 = y_0^* = 4$$

$$y_{n+1}^* = y_n^* + h \cdot f(t_n, y_n^*)$$

$$f(t_0, y_0^*) = \frac{t_0^2 - y_0^*}{2} = \frac{0^2 - 4}{2} = -2$$

$$y_1^* = y_0^* + h \cdot f(t_0, y_0^*) = 4 + 0.05(-2) = 4 + (-0.1) = 3.9$$

$$f(t_1, y_1^*) = \frac{t_1^2 - y_1^*}{2} = \frac{(0.05)^2 - 3.9}{2} = -1.9488$$

$$y_2^* = y_1^* + h \cdot f(t_1, y_1^*) = 3.9 + 0.05(-1.9488) = 3.9 + (-0.0974) = 3.8026$$

$$f(t_2, y_2^*) = \frac{t_2^2 - y_2^*}{2} = \frac{(0.1)^2 - 3.8026}{2} = -1.8963$$

Step 2. We find y_i

$$y_0 = y_0^* = 4$$

$$f(t_0, y_0^*) = f(t_0, y_0) = -2, \; f(t_1, y_1^*) = -1.9488, \; f(t_2, y_2^*) = -1.8963$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}^*)\right]$$

$$y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, y_1^*)\right] = 4 + \frac{0.05}{2}\left[-2 + (-1.9488)\right] = 3.9013$$

$$f(t_1, y_1) = \frac{t_1^2 - y_1}{2} = \frac{(0.05)^2 - 3.9013}{2} = -1.9494$$

$$y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, y_2^*)\right] = 3.9013 + \frac{0.05}{2}\left[-1.9494 + (-1.8963)\right]$$

$$= 3.9013 + \frac{0.05}{2}\left[-3.8457\right] = 3.9013 + (-0.0961) = 3.8052$$

□

Homework of Modified Euler Method

Use Modified Euler method to solve the $D.E.$ and find y_2

$$\frac{dy}{dt} + 2y = 1.3e^{-t}, \quad \text{with } t_0 = 0, y_0 = 0.5, h = 1$$