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المحاضرة الثالثة

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chapter two

Partial Differential Equations (P. D.E)

Partial Differential Equations are Differential Equations in which the unknown function of more than one independent variable.

Types of (P. D.E)

The following some type of (P. D.E):-

1-Order of (P. D.E)

The order of **(P. D.E)** is the highest derivative of equation for example:-

$U_x = U_y$ First-order (p. d. e).

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$ Second -order (p. d. e).

2-The Number of Variables

For example:-

$U_x = U_{tt}$ (two variables x and t).

$$U_x = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}$$ (Three variables t, r and □).

3-Linearity

The **(P. D.E)** is linear or non-linear, is linear **(P. D.E)** if u and whose derivative appear in linear form (non- linear if product two dependent variable or power of this variable greater than one).

For example {the general second L. P. D.E in two variable}

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$…………………(*)

Where A, B, C, D, E, F and G are constant or function of x and y for example

$u_{tt} + e^{-t} u_{xx} = \sin t$ (Linear)

$u_{xx} = yu_{yy}$ (Linear)

$u\, u_x + u_y = 0$ (Non-Linear)

$x\, u_x + y\, u_y + u^2 = 0$ (Non-Linear).

4-Homogeneity

If each term of (P. D.E) contain the unknown function and which derivative is called (H. P. D.E) otherwise is called (non-H. P. D.E), in special case in (*) is homogeneous if [G =0]. Otherwise non-homogeneous.

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$ (H. P. D.E)

Where A, B, C, D, E and F are constant or function of x and y.

Example1

Determine which (L. P. D.E) is, order and dependent or independent variable in following:-

$$1-\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2}$$

Linear second degree u, dependent variable, x and t are independent variable.

$$2-x^2\frac{\partial^3 r}{\partial y^3}=y^3\frac{\partial^2 r}{\partial x^2}$$

Linear 3- degree(r, dependent variable, x and y are independent variable.

$$3-w\frac{\partial^3 w}{\partial y^3}=rst$$

Non-Linear 3- degree(w, dependent variable, r, s and t are independent variable.

$$4-\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 Q}{\partial z^2}=0$$

Linear 2- degree(Q, dependent variable, x, y and z are independent variables, homogeneous.

$$5-(\frac{\partial u}{\partial t})^2+(\frac{\partial u}{\partial x})^2=0$$

Non-Linear 1- degree(u, dependent variable, t and x are independent variables, homogeneous.

Solution of (P. D.E)

A solution of (P. D.E) mean that the value of dependent variable which satisfied the (P. D.E) at all points in given region R.
For Physical Problem, we must be given other conditions at boundary, these are called boundary if these condition are given at t=0 we called them as initial conditions its order.
For a linear homogeneous equation if
$u_1, u_2 \dots u_n$ are n solution then the general solution can be written as (n-th order p. d. e)
$u= c_1 u_1+ c_2 u_2+ \dots + c_n u_n$.

Note i

We can find the solution of (P. D.E) by sequence of integrals as see in the following examples:-

Example2

Find the solution of the following (P. D.E)

$$\frac{\partial^2 z}{\partial x \partial y}=0$$

Solution

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})=0$ By integrate (w. r. to) x gives

$\frac{\partial z}{\partial y}=c(y)$

Where c(y) is arbitrary parametric of y. Also by integrate (w. r. to) y gives

$z=\int c(y)\partial y+c(x)$

Where c(x) is arbitrary parametric of x.

Example3

Find the solution of the following (P. D.E)

$$\frac{\partial^2 z}{\partial x\partial y}=x^2 y$$

Solution

By integrate (w. r. to) x gives

$\frac{\partial z}{\partial y}=\frac{x^3 y}{3}+c(y)$

By integrate (w. r. to) y gives

$$z=\frac{x^3 y^2}{6}+\int c(y)\partial y+c(x)$$

$$z=\frac{x^3 y^2}{6}+F(y)+c(x)$$

Example4

Find the solution of the following (P. D.E)

$\frac{\partial^2 u}{\partial x\partial y}=6x+12y^2$

With boundary condition, $u(1,y)=y^2-2y$, $u(x,2)=5x-5$

Solution

By integrate (w. r. to) x gives

$$\frac{\partial u}{\partial y}=3x^2+12y^2x+c(y)$$

By integrate (w. r. to) y gives

$$u=3x^2y+4y^3x+\int c(y)\partial y+g(x)$$

$\therefore u(x,y)=3x^2y+4y^3x+h(y)+g(x)$

$u(1,y)=3y+4y^3+h(y)+g(1)=y^2-2y$

$h(y)=y^2-4y^3-5y-g(1)$

$\therefore u(x,y)=3x^2y+4y^3x+y^2-4y^3-5y-g(1)+g(x)$

$\therefore u(x,2)=6x^2+32x+4-32-10-g(1)+g(x)=5x-5$

$g(x)=33-27x-6x^2+g(1)$

$$\therefore u(x,y) = 3x^2 y + 4y^3 x + y^2 - 4y^3 - 5y + 33 - 27x - 6x^2$$

Formation of (P. D.E)

A (P. D.E) may formed by a eliminating arbitrary constants or arbitrary function from a given relation and other relation obtained by differentiating partially the given relation.

Note ii

Suppose the following relation:-

$$1 - \frac{\partial z}{\partial x} = z_x = p$$

$$2 - \frac{\partial z}{\partial y} = z_y = q$$

$$3 - \frac{\partial^2 z}{\partial x^2} = z_{xx} = r$$

$$4 - \frac{\partial^2 z}{\partial y^2} = z_{yy} = t$$

$$5 - \frac{\partial^2 z}{\partial x \partial y} = z_{yx} = s$$

Example 5

Form a **Partial Differential Equations** from the following equation:-

$Z = (x - a)^2 + (y - b)^2$……………………………(1)

Solution

$$\frac{\partial z}{\partial x} = z_x = 2(x - a)$$

$$\frac{\partial z}{\partial y} = z_y = 2(y - b)$$

□Eq(1) become

$$Z = (\frac{1}{2} z_x)^2 + (\frac{1}{2} z_y)^2$$

$$4Z = (z_x)^2 + (z_y)^2$$

$$4Z = (p)^2 + (q)^2$$

Example 6

Form a **Partial Differential Equations** from the following equation:-

$Z = f(x^2 + y^2)$……………………………(2)

Solution

$Z_x = 2xf'(x^2 + y^2)$

$Z_y = 2yf'(x^2 + y^2)$

Eq(2) become

$$\frac{z_x}{z_y} = \frac{x}{y},$$

$-x Z_y + y Z_x = 0$

$yp - xq = 0$

Example 7

Form a **Partial Differential Equations** from the following equation:-

$Z= ax+by+a^2 +b^2$.................................. (3).

Solution

$Z_x=a$

$Z_y=b$

Eq(3) become

$Z= x Z_x +y Z_y +(Z_x)^2 +(Z_y)^2$

$Z= x p +y q +(p)^2 +(q)^2$

Example 8

Form a **Partial Differential Equations** from the following equation:-

$v= f(x -ct) +g(x+ct)$

Solution

$v_x= f'(x -ct)+g'(x+ct)$

$v_t= -cf'(x -ct)+cg'(x+ct)$

$v_{xx}= f''(x -ct)+g''(x+ct)$

$v_{tt}= c^2f''(x -ct)+c^2g''(x+ct)$

$v_{tt}= c^2 [f''(x -ct)+g''(x+ct)]$

□ $v_{tt}= c^2 v_{xx}$, or

$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}$ One dimensional Wave equation