



Al-Mustaqbal University
College of Science
Artificial Intelligence Department
First Stage



جامعة المستقبل
AL MUSTAQBAL UNIVERSITY

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Lecture (3)

LIMITS

المادة : الرياضيات

المرحلة : الاولى

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<https://classroom.google.com/c/ODM3NjM4NTg2MzY3?cjc=k7kgjyvz>



Limit of a Function:.

If f is a function, then we say:

A is the limit of $f(x)$ as x approaches a if the value of $f(x)$ gets arbitrarily close to A as x approaches a . This is written in mathematical notation as:

$$\lim_{x \rightarrow a} f(x) = A$$

For example:

$$\lim_{x \rightarrow 3} x^2 = 9$$

$$\lim_{x \rightarrow 0} x = 0$$

$$\lim_{x \rightarrow 2} 4 = 4$$

The limit law:.

If L , M , c and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.



5. *Quotient Rule:* $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \rightarrow c} (f(x))^{r/s} = L^{r/s}$$

Example:.

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3) = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3$ Sum and Difference Rules
 $= c^3 + 4c^2 - 3$ Product and Multiple Rules

(b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)}$ Quotient Rule
 $= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5}$ Sum and Difference Rules
 $= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Power or Product Rule

(c) $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)}$ Power Rule with $r/s = 1/2$
 $= \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3}$ Difference Rule
 $= \sqrt{4(-2)^2 - 3}$ Product and Multiple Rules
 $= \sqrt{16 - 3}$
 $= \sqrt{13}$



Indeterminate quantities :

$\frac{0}{0}$,	$\frac{\infty}{\infty}$,	$0 \cdot \infty$,	$\infty - \infty$,	∞^0	,	0^0
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Example:. $\lim_{x \rightarrow -2} \frac{-2x-4}{x^3+2x^2}$

Solution: $\frac{-2 \cdot -2 - 4}{(-2)^3 + 2(-2)^2} = \frac{0}{0}$, then the solution

$$\lim_{x \rightarrow -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \rightarrow -2} \frac{-2}{x^2} = \frac{-2}{(-2)^2} = \frac{-2}{4} = \frac{-1}{2}$$

Example:. $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} \quad \frac{0}{0}$

Solution: $\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 2+2=4$

Example:. $\lim_{x \rightarrow 0} \frac{x}{1-\sqrt{1-x}} \quad \frac{0}{0}$

Solution: $\lim_{x \rightarrow 0} \frac{x(1+\sqrt{1-x})}{(1-\sqrt{1-x})(1+\sqrt{1-x})} = \lim_{x \rightarrow 0} \frac{x(1+\sqrt{1-x})}{1-(1-x)}$

$$= \lim_{x \rightarrow 0} \frac{x(1+\sqrt{1-x})}{1-1+x} = \lim_{x \rightarrow 0} \frac{x(1+\sqrt{1-x})}{x} = \lim_{x \rightarrow 0} (1+\sqrt{1-x}) = 2$$

Example:. $\lim_{x \rightarrow 0} \frac{\sqrt{x^2+100}-10}{x^2} \quad \frac{0}{0}$

Solution: $\lim_{x \rightarrow 0} \frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)}$

$$= \lim_{x \rightarrow 0} \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{x^2+100}+10)}$$



$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} = \frac{1}{\sqrt{0 + 100} + 10} = \frac{1}{10 + 10} = \frac{1}{20}$$

Limits of trigonometric functions:

If a is constant.

$\lim_{x \rightarrow 0} \sin x = 0$	$\lim_{x \rightarrow 0} \cos x = 1$	$\lim_{x \rightarrow 0} \tan x = 0$
$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	$\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$	$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$
$\lim_{x \rightarrow 0} \frac{\tan ax}{ax} = 1$	$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$	

Example: Find

$$1. \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{1}{5} \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \frac{1}{5} \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{x} * \frac{2}{2}$$

$$= \frac{2}{5} \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = \frac{2}{5} * 1 = \frac{2}{5}$$

$$2. \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{\sin 2x \frac{2x}{2x}}{\sin 3x \frac{3x}{3x}} = \lim_{x \rightarrow 0} \frac{2x \frac{\sin 2x}{2x}}{3x \frac{\sin 3x}{3x}}$$

$$= \frac{2}{3} \quad \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}} = \frac{2}{3} \quad \frac{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x}}{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x}} = \frac{2}{3} * \frac{1}{1} = \frac{2}{3}$$

$$3. \quad \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} = \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} = \frac{1}{1} = 1$$

$$4. \quad \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta^2} = \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right)^2 = (1)^2 = 1$$

$$\text{or} \quad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} * \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 * 1 = 1$$



Infinite limits [as $x \rightarrow \pm \infty$] ∴.

The basic facts to be verified by applying the formal definition are.

$$\lim_{x \rightarrow \pm \infty} k = k \quad \text{and} \quad \lim_{x \rightarrow \pm \infty} \frac{1}{x} = 0$$

$$\text{Example} \therefore \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) = \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} = 5 + 0 = 5$$

$$\text{Example} \therefore \lim_{x \rightarrow -\infty} \frac{\pi \sqrt{3}}{x^2} = \lim_{x \rightarrow -\infty} \pi \sqrt{3} \frac{1}{x} * \frac{1}{x} = \lim_{x \rightarrow -\infty} \pi \sqrt{3} *$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} * \lim_{x \rightarrow -\infty} \frac{1}{x} = \pi \sqrt{3} * 0 * 0 = 0$$

$$\begin{aligned} \text{Example} \therefore \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{\frac{5x^2}{x^2} + \frac{8x}{x^2} - \frac{3}{x^2}}{\frac{3x^2}{x^2} + \frac{2}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} \\ &= \frac{5 + 0 + 0}{3 + 0} = \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \text{Example} \therefore \lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{11x}{x^3} + \frac{2}{x^3}}{\frac{2x^3}{x^3} - \frac{1}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}} \\ &= \frac{0 + 0}{2 - 0} = \frac{0}{2} = 0 \end{aligned}$$

$$\text{Example} \therefore \lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right) = \sin\left(\frac{1}{\infty}\right) = \sin(0) = 0$$



H.W: Find .

1. $\lim_{x \rightarrow 2} \frac{x+3}{x+6}$	$(\frac{5}{8})$	2. $\lim_{x \rightarrow 2/3} 3x(2x-1)$	$(\frac{2}{3})$
3. $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$	$(\frac{1}{10})$	4. $\lim_{x \rightarrow -3} \frac{x+3}{x^2+4x+3}$	$(\frac{-1}{2})$
5. $\lim_{t \rightarrow 1} \frac{t^2+t-2}{t^2-1}$	$(\frac{3}{2})$	6. $\lim_{u \rightarrow 1} \frac{u^2-1}{u^3-1}$	$(\frac{2}{3})$
7. $\lim_{x \rightarrow 0} (2 \sin x - 1)$	(-1)	8. $\lim_{x \rightarrow 0} \sec x$	(1)
9. $\lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x)$	(-1)	10. $\lim_{x \rightarrow \pi} \sqrt{x+4} \cos(x+\pi)$	$(\sqrt{\pi+4})$
11. $\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3 \cos x}$	$(\frac{1}{3})$	12. $\lim_{x \rightarrow 0} \frac{\sin(1-\cos x)}{1-\cos x}$	(1)
13. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$	$(\frac{1}{2})$	14. $\lim_{\theta \rightarrow 0} \theta \cos \theta$	(0)
15. $\lim_{x \rightarrow \pi} \sin(x - \sin x)$	(0)	16. $\lim_{t \rightarrow 0} \cos(\frac{\pi}{\sqrt{19-3 \sec 2t}})$	$(\frac{1}{\sqrt{2}})$
17. $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$	(2)	18. $\lim_{x \rightarrow \infty} (\frac{1-x^3}{x^2+7x})^5$	(∞)
19. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$	(0)	20. $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$	(1)
21. $\lim_{x \rightarrow \infty} \frac{3-x}{\sqrt{4x^2+25}}$	$(1/2)$	22. $\lim_{x \rightarrow \infty} (\frac{3}{x^2} - \cos \frac{1}{x})(1 + \sin \frac{1}{x})$	(-1)