College of Sciences
Artifactual intelligent science
Department

Lecture 6:

Crisp Relations and Fuzzy Relations

Subject: Fuzzy Logic

Level: Second

Semester: second

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Google Class Room

Introduction to Relations

In mathematics, computer science, database systems, and fuzzy logic, a relation describes a connection between elements. It tells us which element is linked to which other element. A relation may connect elements from two different sets, or it may connect elements inside the same set.

A good way to think about a relation is this: if we can ask a question such as "Is x related to y?", then the collection of all true answers forms a relation.

Simple example:

Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$.
One possible relation is $R = \{(1,4), (2,5), (3,4)\}$.

This means 1 is related to 4, 2 is related to 5, and 3 is related to 4. Each ordered pair (x, y) states that x is connected to y under the relation R.

Mathematical Definition of a Relation

A relation from set A to set B is any subset of the Cartesian product $A \times B$. The Cartesian product contains all possible ordered pairs formed by taking the first element from A and the second element from B.

$$\mathbf{R \subseteq A \times B}$$

Example:
If $A = \{1, 2\}$ and $B = \{3, 4\}$, then
$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

Now a relation can choose some of these pairs. For example, $R = \{(1,3), (2,4)\}$ is a valid relation from A to B. Notice that a relation does not need to contain all pairs. It only contains the pairs that satisfy the rule we choose.

Common Ways to Represent a Relation

Students often understand relations more easily when they see them in different forms. The same relation can be written in three common ways.

1. Ordered pairs

This is the most direct form. We list every pair that belongs to the relation, such as R = {(1,2), (2,3), (3,1)}.

2. Relation matrix

A matrix is useful when the set is small. Rows represent the first element and columns represent the second element. If a pair belongs to the relation, we write 1; otherwise we write 0.

R	1	2	3
1	0	1	0
2	0	0	1
3	1	0	0

The matrix above represents the relation R = {(1,2), (2,3), (3,1)}.

3. Directed graph

In a directed graph, elements are drawn as nodes and relations are drawn as arrows. For example, the relation {(1,2), (2,3), (3,1)} can be visualized as arrows $1 \to 2$, $2 \to 3$, and $3 \to 1$.

Crisp Relations

A crisp relation is the classical form of a relation used in ordinary set theory. A pair either belongs to the relation or does not belong to it. There is no partial belonging.

$$\mu R(x, y) \in \{0, 1\}$$

$\mu R(x, y) = 1$ means the pair (x, y) belongs to the relation.
$\mu R(x, y) = 0$ means the pair (x, y) does not belong to the relation.

This kind of relation is suitable when the rule is exact, such as “greater than”, “equal to”, or “is a divisor of”.

Example: relation “x is greater than y” on A = {1, 2, 3}
The valid pairs are {(2,1), (3,1), (3,2)}.

Here the pair (2,1) belongs because $2 > 1$. The pair (1,3) does not belong because 1 is not greater than 3. So membership is only 0 or 1.

Fuzzy Relations

A fuzzy relation extends the crisp idea by allowing degrees of relation. Instead of saying a pair is either fully in or fully out, fuzzy logic allows partial membership.

$$\mu R(x, y) \in [0, 1]$$

This is important in real life because many relationships are not perfectly black or white. For example, two people may be similar in height to degree 0.9, but similar in age to degree 0.4.

Example: similarity of students' heights

Student A	Student B	Similarity degree
Ahmed	Ali	0.9

Student A	Student B	Similarity degree
Ahmed	Omar	0.4
Ali	Omar	0.6

This table does not say that two students are either completely similar or not similar at all. It expresses how strong the similarity is. That is the key difference between a fuzzy relation and a crisp relation.

Difference Between Crisp and Fuzzy Relations

Feature	Crisp relation	Fuzzy relation
Membership values	Only 0 or 1	Any value between 0 and 1
Type of decision	Exact / binary	Gradual / partial
Best for	Precise mathematical rules	Similarity, preference, uncertainty
Example	$x > y$	x is similar to y

Relations on a Single Set

A relation on a single set means that the first and second elements of each ordered pair come from the same set. This is the kind of relation used when we study the properties reflexive, symmetric, and transitive.

Example:

Let $A = \{1, 2, 3\}$. A relation on A could be $R = \{(1,1), (2,2), (3,3), (1,2)\}$.

Because both entries of every pair come from the same set A, this is a relation on a single set.

Properties of Relations

Relations on a single set can have several important properties. These properties help us understand the structure of relations.

The most common properties are:

1. ***Reflexive***
2. ***Symmetric***
3. ***Transitive***

1. Reflexive Relation

A relation R on a set A is reflexive if every element is related to itself. That means all diagonal pairs must be included.

For every x ∈ A, (x, x) ∈ R

How to test reflexivity:

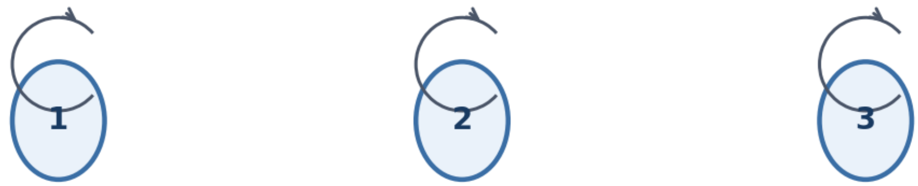
- List all elements of the set.
- Check whether each self-pair appears in the relation: (1,1), (2,2), (3,3), and so on.
- If even one self-pair is missing, the relation is not reflexive.

Example:

A = {1, 2, 3}
R = {(1,1), (2,2), (3,3), (1,2)}

This relation is reflexive because all three self-pairs (1,1), (2,2), and (3,3) are present. The extra pair (1,2) does not affect reflexivity.

Reflexive Relation



In the graph, each node has a loop to itself, which visually confirms reflexivity.

2. Symmetric Relation

A relation R on a set A is symmetric if whenever one arrow goes from x to y, another arrow goes back from y to x.

If (x, y) ∈ R, then (y, x) ∈ R

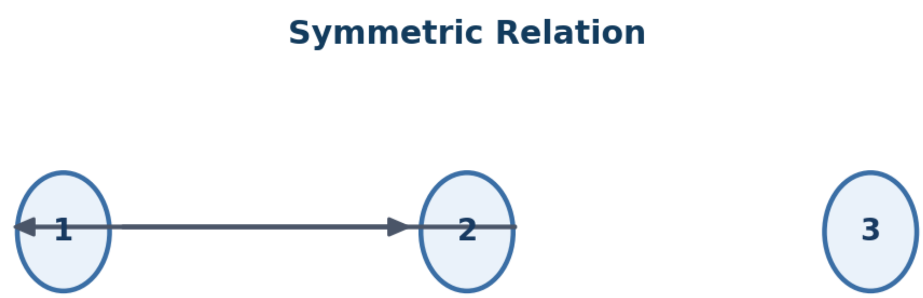
How to test symmetry:

- Take any pair (x, y) in the relation.
- Look for the reverse pair (y, x).
- If one reverse pair is missing, the relation is not symmetric.

Example:
R = {(1,2), (2,1), (2,3), (3,2)}

This relation is symmetric because every pair has its reverse. The pair (1,2) is matched by (2,1), and the pair (2,3) is matched by (3,2).

A useful real-life example is the relation "is a classmate of" in the same class. If Ahmed is a classmate of Ali, then Ali is a classmate of Ahmed.



The opposite arrows show that the relation goes both ways.

3. Transitive Relation

A relation R on a set A is transitive if relation can be passed through a middle element. In other words, if x is related to y and y is related to z, then x must also be related to z.

If (x, y) ∈ R and (y, z) ∈ R, then (x, z) ∈ R

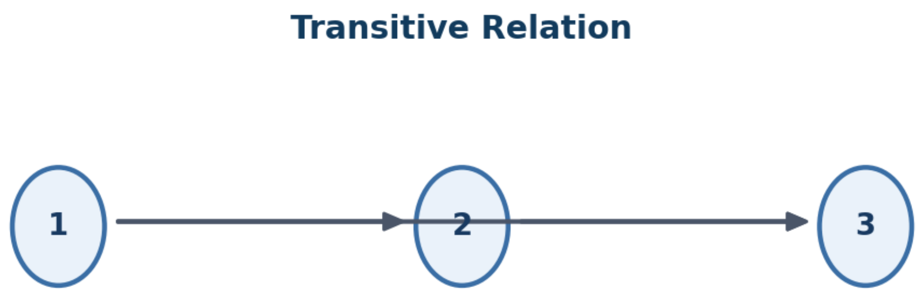
How to test transitivity:

- Find two pairs that connect through a middle element, such as (1,2) and (2,3).
- Check whether the direct pair (1,3) also appears.
- If a needed direct pair is missing, the relation is not transitive.

Example:
R = {(1,2), (2,3), (1,3)}

This relation is transitive because the presence of (1,2) and (2,3) forces (1,3), and that pair is already included.

A common real-life example is the relation “is older than”. If Ali is older than Omar and Omar is older than Hasan, then Ali is older than Hasan.



The direct arrow 1→3 is what makes the shown example transitive.

Homework

Let $A = \{1,2,3\}$and let

$$R = \{(1,1), (2,2), (1,2), (2,3)\}.$$

Determine whether the relation $R$is:

Reflexive
Symmetric
Transitive