



Definition

A matrix is a set of real or complex numbers (or elements) arranged in rows and columns to form rectangular array. A matrix having m rows and n columns is called $(m \times n)$ matrix and is referred to as having order $(m \times n)$. A matrix is in dictating by writing the array within brackets.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A horizontal line of elements is called row, and a vertical line is called a column.

For example:

$$\begin{bmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{bmatrix} \text{ is } (2 \times 3) \text{ matrix}$$

2 the numbers of rows and 3 the number of columns.

(Discussion)



Types of Matrices:

-Row matrix: consists of 1 row only, for example $[4 \ 3 \ 7 \ 2]$ is a row matrix of order (1×4) .

-Column matrix: consists of 1 column only, for example

$$\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \text{ is a column matrix of order } (3 \times 1).$$

-Square matrix: is a matrix in which the number of rows (**m**) equals the number of columns (**n**) for example

$$S = \begin{bmatrix} 2 & 0 & 5 \\ 7 & 8 & 7 \\ 6 & 7 & 5 \end{bmatrix}$$

-Rectangular matrix: A matrix of any size (**m** × **n**) and this includes square matrices as a special case.

-Diagonal matrix: is a square matrix with all elements zero except those on the main diagonal. For example

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$



-Unit matrix: is a diagonal matrix in which the elements on the main diagonal are all unity. For example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

-Null (zero) matrix: is one whose elements are zero. For example

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

-Vector matrix: is a matrix with only one row or column. Its entries are called the component of the vector.

Some Operations on Matrices:

1- Equality of matrices:

Two matrices A and B are equal if and only if they have the same size and corresponding entries are equal, matrices that are not equal are called different.

Example: let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$

$A=B$ if and only if: $a_{11} = 4, a_{12} = 0, a_{21} = 3, a_{22} = -1$

Example: If $A = \begin{bmatrix} 2 & 3 & a \\ b & 6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 9 \\ -3 & 6 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 9 \\ -3 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Then $A = B$ if $a = 9$ and $b = -3$ but $A \neq C$ and $B \neq C$.



2- Addition and Subtraction of Matrices:

To be added or subtracted, two matrices must be of the same order. The sum or difference is then determined by adding or subtracting corresponding elements. For example:

$$\begin{bmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 & 12 \\ 9 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 7 & 1 \\ 2 & 10 & -5 \end{bmatrix} = \begin{bmatrix} 6-3 & 5-7 & 12-1 \\ 9-2 & 4-10 & 8+5 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 11 \\ 7 & -6 & 13 \end{bmatrix}$$

3- Multiplication of Matrices:

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second, for example:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then:

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}$$



Example: If $A = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}$, find $A \cdot B$

Sol:

$$A \cdot B = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 * 8 + 7 * 5 + 6 * 9 \\ 2 * 8 + 3 * 5 + 1 * 9 \end{bmatrix} = \begin{bmatrix} 32 + 35 + 54 \\ 16 + 15 + 9 \end{bmatrix} = \begin{bmatrix} 121 \\ 40 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{bmatrix}$, find $A \cdot B$

Sol:

$$A \cdot B = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 8 + 10 & 4 + 25 & 3 + 40 & 1 + 30 \\ 16 + 14 & 8 + 35 & 6 + 56 & 2 + 42 \\ 24 + 8 & 12 + 20 & 9 + 32 & 3 + 24 \end{bmatrix}$$
$$= \begin{bmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{bmatrix}$$

Note: If A is an $(m \times n)$ matrix and B is $(n \times m)$ matrix, then products $A \cdot B$ and $B \cdot A$ are possible.



Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$, find $A.B$ and $B.A$

Sol:

$$A.B = \begin{bmatrix} 7 + 16 + 27 & 10 + 22 + 36 \\ 28 + 40 + 54 & 40 + 55 + 72 \end{bmatrix} = \begin{bmatrix} 50 & 68 \\ 122 & 167 \end{bmatrix}$$

$$B.A = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 7 + 40 & 14 + 50 & 21 + 60 \\ 8 + 44 & 16 + 55 & 24 + 66 \\ 9 + 48 & 18 + 60 & 27 + 72 \end{bmatrix}$$

$$= \begin{bmatrix} 47 & 64 & 81 \\ 52 & 71 & 90 \\ 57 & 78 & 99 \end{bmatrix}$$

Note: $A.B \neq B.A$ Multiplication is not commutative.

Properties of Matrix Operations:

If A and B, C are $(m \times n)$ matrices, O is zero matrix and K, R are any scalars then:

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + O = A$, $A + (-A) = O$
- $K(RA) = (KR)A$, $(K + R)A = KA + RA$
- $IA = A$, $OA = O$, $RO = O$



Transpose of Matrix:

If the rows and columns of matrix are interchanged, then the new matrix is called the transpose of the original matrix. If A^T is the transpose of matrix A , then

$A \neq A^T$, For example:

$$A = \begin{bmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 3 & 7 \\ 1 & 5 \end{bmatrix}$, find $A.B$ and $(A.B)^T$.

Sol:

$$A.B = \begin{bmatrix} 35 & 79 \\ 20 & 32 \end{bmatrix}, \quad (A.B)^T = \begin{bmatrix} 35 & 20 \\ 79 & 32 \end{bmatrix}$$

Special Matrices:

Square matrix is a matrix of order ($m \times m$). A square matrix is **symmetric** if $a_{ij} = a_{ji}$ means $A = A^T$ for example

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{bmatrix}$$



A square matrix is **skew-symmetric** if $a_{ij} = -a_{ji}$ means $A \neq A^T$ for example

$$A = \begin{bmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -9 \\ 5 & 9 & 0 \end{bmatrix}$$

Example: Given that $A = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix}$ determine A^T and $A.A^T$.

Sol:

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{bmatrix}$$

$$A.A^T = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 16 + 4 + 36 & 4 + 16 + 42 \\ 4 + 16 + 42 & 4 + 64 + 49 \end{bmatrix} = \begin{bmatrix} 56 & 62 \\ 62 & 117 \end{bmatrix}$$

Determinant of a Square Matrix:

The determinant of a square matrix is the determinant having the same elements as those of the matrix. For example

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{bmatrix} \quad \text{then the det of A is given by:}$$

$$\begin{aligned} |A| &= 5(42 - 12) - 2(0 - 24) + 1(0 - 48) = 5(30) - 2(-24) + 1(-48) \\ &= 150 + 48 - 48 = 150 \end{aligned}$$



Note that the transpose of $A^T = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 7 \end{bmatrix}$

And the determinate of A^T is

$$|A^T| = 5(42 - 12) - 0(14 - 4) + 8(6 - 6) = 5(30) = 150$$

Note that: $\det(A) = \det(A^T)$.

Cofactors:

If A is square matrix, the determinates of its element will be:

If A is square matrix, the determinates of its element will be:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\det(A) = 2(0 - 24) - 3(0 - 6) + 5(16 - 1) = 45$$

The minor of element 2 is $+ \begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix} = 0 - 24 = -24$

Similarly the cofactor of element 3 is $- \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = -(0 - 6) = 6$



The cofactor of element 5 is $+ \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = +(16 - 1) = 1$

The cofactor of element 4 is $- \begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} = -(0 - 20) = 20$

The cofactor of element 1 is $+ \begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = +(0 - 5) = -5$

The cofactor of element 6 is $- \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -(8 - 3) = -5$

The cofactor of element 1 is $+ \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = +(18 - 5) = 13$

The cofactor of element 4 is $- \begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} = -(12 - 20) = 8$

The cofactor of element 0 is $+ \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = +(2 - 12) = -10$

The cofactor matrix C is

$$C = \begin{bmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{bmatrix}$$



And the transpose of C is

$$C^T = \begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix}$$



Where C^T is called the adjoint of matrix $A = adj A$.



Inverse of a Square Matrix:

If $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$ then the inverse of A is A^{-1} and given by:

$$A^{-1} = \frac{\mathit{adj} A}{\det A}$$

$$\det A = |A| = 2(0 - 24) - 3(0 - 6) + 5(16 - 1) = 45$$

The cofactor matrix C is:

$$C = \begin{bmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{bmatrix} \text{ and } C^T = \mathit{adj} A = \begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{\mathit{adj} A}{\det A} = \frac{1}{45} \begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix} = \begin{bmatrix} -24/45 & 20/45 & 13/45 \\ 6/45 & -5/45 & 8/45 \\ 15/45 & -5/45 & -10/45 \end{bmatrix}$$



Example: Find the inverse of the given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{bmatrix}$$

Sol:

$$\det A = |A| = 1(2 - 0) - 2(8 - 30) + 3(0 - 6) = 28$$

The cofactor of element 1 is $+(2 - 0) = 2$

The cofactor of element 4 is $-(4 - 0) = -4$

The cofactor of element 6 is $+(10 - 3) = 7$

The cofactor of element 2 is $-(8 - 30) = 22$

The cofactor of element 1 is $+(2 - 18) = -16$

The cofactor of element 0 is $-(5 - 12) = 7$

The cofactor of element 3 is $+(0 - 6) = -6$

The cofactor of element 5 is $-(0 - 12) = 12$

The cofactor of element 2 is $+(1 - 8) = -7$

The cofactor matrix C is

$$C = \begin{bmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{bmatrix} \quad \text{and} \quad \text{adj } A = C^T = \begin{bmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{bmatrix}$$