Al-Mustaqbal University
College of Engineering Technology
Department of Cyber Security Techniques Engineering
Class: 1st
Subject: Math
Lecturer: Dr. Hussein Ali Ameen
1st term – Lecture: 8+9 - Differentiation

Definition

The derivative of a function $f(x)$ with respect to the variable x is the function $f(x)$ whose value at x is:

$$\acute{f}(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

Provided this limit exists.

Differentiation Rules:

1-Derivative of a constant function

If $y=f(x)=c$ where c is a constant then

$$\frac{dy}{dx} = \acute{f}(x) = 0$$

Example1: $f(x)=1 \; then \; \frac{dy}{dx} = \acute{f} = 0$

Example2: $f(x)=25 \; then \; \frac{dy}{dx} = \acute{f} = 0$

Example3: $f(x)=-1 \; then \; \frac{dy}{dx} = \acute{f} = 0$

Example4: $f(x)=1000 \; then \; \frac{dy}{dx} = \acute{f} = 0$

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2-Power rule for positive integers:

If n is a positive integer, and $\boldsymbol{y = f(x) = x^n}$;then

$$\acute{f}(x) = \frac{dy}{dx} = n\, x^{n-1}$$

Example: $y = f(x) = x^3 \quad then \quad \acute{f}(x) = \frac{dy}{dx} = 3\, x^2$

Example: $y = f(x) = x^8 \quad then \quad \acute{f}(x) = \frac{dy}{dx} = 8\, x^7$

$f(x) = x^2$	$f^{/}(x) = nx^{n-1}$	٢- مشتقة المتغير $f(x) = x^n$

3-Derivative constant multiple rule:

If u is a differentiable function of x $f(x) = c\, u(x)$, and c is a constant, then

$$\frac{dy}{dx} = \acute{f}(x) = c\acute{u}(x)$$

Example: $f(x) = 5x^7 \quad then \ \frac{dy}{dx} = 5 * 7\, x^6 = 35\, x^6$

Example: $f(x) = 2x^4 \quad then \ \frac{dy}{dx} = 2 * 4\, x^3 = 8\, x^3$

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1) $f(x) = x \Longrightarrow f'(x) = 1$

2) $f(x) = x^2 \Longrightarrow f'(x) = 2x$

3) $f(x) = x^3 \Longrightarrow f'(x) = 3x^2$

4) $f(x) = x^4 \Longrightarrow f'(x) = 4x^3$

5) $f(x) = x^5 \Longrightarrow f'(x) = 5x^4$

6) $f(x) = 2x \Longrightarrow f'(x) = 2$

7) $f(x) = 3x^2 \Longrightarrow f'(x) = 6x$

8) $f(x) = 5x^3 \Longrightarrow f'(x) = 15x^2$

9) $f(x) = x^{-2} \Longrightarrow f'(x) = -2x^{-3}$

10) $f(x) = x^{-3} \Longrightarrow f'(x) = -3x^{-4}$

11) $f(x) = x^{-4} \Longrightarrow f'(x) = -4x^{-5}$

12) $f(x) = 3x^{-5} \Longrightarrow f'(x) = -15x^{-6}$

مشتقة الجذور

1) $f(x) = \sqrt{x} = x^{\frac{1}{2}} \Longrightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

2) $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Longrightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

3) $f(x) = \sqrt[4]{x} = x^{\frac{1}{4}} \Longrightarrow f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$

4) $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}} \Longrightarrow f'(x) = \frac{1}{5}x^{-\frac{4}{5}}$

5) $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}} \Longrightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

6) $f(x) = \sqrt[5]{x^3} = x^{\frac{3}{5}} \Longrightarrow f'(x) = \frac{3}{5}x^{-\frac{2}{5}}$

7) $f(x) = \sqrt[3]{x^{-2}} = x^{-\frac{2}{3}} \Longrightarrow f'(x) = -\frac{2}{3}x^{\frac{1}{3}}$

Note

Derivative of polynomial functions: Differentiate each term of the polynomial separately and keep the plus/minus signs

$f(x) = h(x) \mp g(x) \Longrightarrow f^{/}(x) = h^{/}(x) \mp g^{/}(x)$

1) $f(x) = 3x^5 + 7x \Longrightarrow f^{/}(x) = 15x^4 + 7$
2) $f(x) = 3x^4 - 4x^2 + 6 \Longrightarrow f^{/}(x) = 12x^3 - 8x$
3) $f(x) = 2x^2 + \frac{1}{2}x \Longrightarrow f^{/}(x) = 4x + \frac{1}{2}$
4) $f(x) = \frac{1}{2}x^2 - \frac{4}{3}x^3 + 9 \Longrightarrow f^{/}(x) = x - 4x^2$
5) $f(x) = \frac{1}{5}x^{-2} - \frac{2}{7}x^{-3} + 9 \Longrightarrow f^{/}(x) = \frac{-2}{5}x^{-3} + \frac{6}{7}x^{-4}$

4-Derivative Product rule:

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

٤- مشتقة حاصل ضرب دالتين = الاولى في مشتقة الثانية + الثانية في مشتقة الاولى

$f(x) = g(x) . h(x) \Longrightarrow f'(x) = g(x) . h'(x) + h(x) . g'(x)$

Example: Find the derivative of $y = (x^2).(x^3)$

Sol:

$$\frac{dy}{dx} = (x^2).3x^2 + (x^3).(2x)$$

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Example: Find the derivative of $y = (x^2 + 8x)(x^3 - 1)$

Sol:

$$\frac{dy}{dx} = (x^2 + 8x).3x^2 + (x^3 - 1).(2x + 8)$$

Examples:

Find the derivative of the $f(x) = (x^4 - x^2 + 1)(5x^6 - 3x)$

$f(x) = (x^4 - x^2 + 1)(5x^6 - 3x)$
$f'(x) = (x^4 - x^2 + 1)(30x^5 - 3) + (5x^6 - 3x)(4x^3 - 2x)$

$x = 2$ عند $f(x) = (4 - x)(x^2 + 3)$

$f(x) = (4 - x)(x^2 + 3)$
$f'(x) = (4 - x)(2x) + (x^2 + 3)(-1)$
$f'(2) = (4 - 2)(2(2)) + (2^2 + 3)(-1) = 2(4) + (7)(-1) = 8 - 7 = 1$

5- Derivative quotient rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient $\frac{u}{v}$ is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

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٥- مشتقة حاصل قسمة دالتين = $\frac{\text{المقام في مشتقة البسط – البسط في مشتقة المقام}}{\text{المقام}^2}$

$$f(x) = \frac{g(x)}{h(x)} \longrightarrow f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{(h(x))^2}$$

Example: Find the derivative of $y = \frac{x^2}{x^3}$

Sol:

$$\frac{dy}{dx} = \frac{(x^3).(2x)-(x^2).3x^2}{(x^3)^2} = \frac{2x^4-3x^4}{x^6}$$

Example: Find the derivative of $y = \frac{t-t^2}{t+4}$

Sol:

$$\frac{d}{dx}\left(\frac{t-t^2}{t+4}\right) = \frac{(t+4).(1-2t)-(t-t^2).1}{(t+4)^2} = \frac{(t+4)(1-2t)-(t-t^2)]}{(t+4)^2}$$

Example: Find the derivative of f (x) = $\frac{x^3+1}{x^4+1}$ at x=1

$$f'(x) = \frac{(x^4+1)(3x^2)-(x^3+1)(4x^3)}{(x^4+1)^2}$$

$$f'(1) = \frac{(1^4+1)(3\times1^2)-(1^3+1)(4\times1^3)}{(1^4+1)^2} = \frac{2\times3-2\times4}{2^2} = \frac{6-8}{4} = \frac{-2}{4} = \frac{-1}{2}$$

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Example: Find the derivative of f (x) = $\frac{4-5x}{x^4+x+1}$ at x= -1

$$f'(x) = \frac{(x^2+x+1)(-5)-(4-5x)(2x+1)}{(x^2+x+1)^2}$$

$$f'(-1) = \frac{((-1)^2+(-1)+1)(-5)-(4-5(-1))(2(-1)+1)}{((-1)^2+(-1)+1)^2}$$

$$f'(-1) = \frac{(1-1+1)(-5)-(4+5)(-2+1)}{(1-1+1)^2} = \frac{-5-(9)(-1)}{1} = -5+9 = 4$$

مشتق القوس مرفوعة الى أس = الاس في القوس مرفوع الى الأس - 1 في مشتقة داخل القوس

$f(x) = [h(x)]^n \Longrightarrow f'(x) = n[h(x)]^{n-1}(h'(x))$

$f(x) = (1-x)^3$

$f'(x) = 3(1-x)^2(-1) = -3(1-x)^2$

$f(x) = (x^3+x^2+x+1)^5$

$f'(x) = 5(x^3+x^2+x+1)^4(3x^2+2x+1)$

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Second and Higher- Order Derivatives:

If $f(x)$ is a given function then

$\frac{dy}{dx} = \acute{f}(x)$ *is first derivaive of y .*

$\frac{d^2y}{dx^2} = \acute{\acute{f}}(x)$ *is second derivaive of y .*

$\frac{d^3y}{dx^3} = \acute{\acute{\acute{f}}}(x)$ *is third derivative of y*. And so on…

Then, in general:

$$\frac{d^n y}{dx^n} = f^n(x) = y^n$$

Example: If $y = (x^2 + 2x + 3)^2$, *find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$

Sol:

$$\frac{dy}{dx} = \acute{y} = 2 . (x^2 + 2x + 3)(2x + 2)$$

$$\frac{d^2y}{dx^2} = 2 \left[(x^2 + 2x + 3) . 2 + (2x + 2) . (2x + 2) \right]$$

$$\frac{d^2y}{dx^2} = 4(x^2 + 2x + 3) + 2(2x + 2)^2$$

Example:

If $y = (x^2 + 2x + 3)^2$, $find\ \frac{dx}{dy}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$

Sol:

$$\frac{dx}{dy} = 6x^2 - 8x + 6$$

$$\frac{d^2y}{dx^2} = 12x$$

$$\frac{d^3y}{dx^3} = 12$$

$$\frac{d^4y}{dx^4} = 10$$

Example:

Find the value of f(x), and f(x) $x = 2$ عند $f''(x)$, $f'(x)$ جد $f(x) = (x^2 - 3)^4$

$$f'(x) = 4(x^2 - 3)^3(2x) = 8x(x^2 - 3)^3$$

$$f'(2) = 8(2)(2^2 - 3)^3 = 16(1)^3 = 16$$

$$f''(x) = 8x(3)(x^2 - 3)^2(2x) + (x^2 - 3)^3(8)$$

$$f''(x) = 48x^2(x^2 - 3)^2 + 8(x^2 - 3)^3$$

$$f''(2) = 48(2)^2(2^2 - 3)^2 + 8(2^2 - 3)^3 = 192 + 8 = 200$$

Note

ملاحظة / مشتقة الجذر التربيعي = مشتقة داخل الجذر على ٢ في الجذر

$f(x) = \sqrt{} \Longrightarrow f'(x) = \dfrac{\text{مشتقة داخل الجذر}}{2\sqrt{}}$

جد $f'(x)$ $f(x) = \sqrt{x^3 + x^2 - 5}$

$$f'(x) = \frac{3x^2+2x}{2\sqrt{x^3+x^2-5}}$$

جد $f'(x)$ $f(x) = \sqrt{x^2 - 2x + 1}$

$$f'(x) = \frac{2x-2}{2\sqrt{x^2-2x+1}}$$

جد y'' , y' $y = x^4 + 5x^3 + 3$

$y' = 4x^3 + 15x^2$
$y'' = 12x^2 + 30x$

Find the value of f(x) جد $f''(-1)$ ، $f''(x)$ ، $f'(x)$ $f(x) = 2x^3 + 4 + \frac{3}{x}$

$f(x) = 2x^3 + 4 + 3x^{-1}$

$f'(x) = 6x^2 - 3x^{-2} \Longrightarrow f'(x) = 6x^2 - \frac{3}{x^2}$

$f''(x) = 12x + 6x^{-3} \Longrightarrow f''(x) = 12x + \frac{6}{x^3}$

$f''(-1) = 12(-1) + \frac{6}{(-1)^3} = -12 - 6 = -18$

Find the value of f(x) $\quad x = 0$ at $f(x) = \frac{1}{\sqrt{2x+1}}$

$$f(x) = (2x+1)^{\frac{-1}{2}}$$

$$f^{/}(x) = \frac{-1}{2}(2x+2)^{\frac{1}{2}}(2) = -\sqrt{2x+1}$$

$$f^{/}(0) = -\sqrt{2(0)+1} = -\sqrt{1} = -1$$

Home Work

EX/ Find $f^{/}$ of functions

(1) $f_{(X)} = (\frac{x}{x+1})^4$ >>>>>> when x = 1

(2) $f_{(X)} = x + \frac{3}{x^2+1}$ >>>>> when x = -1

(3) $f_{(X)} = (x^3 + 3x^2 - 3)^{\frac{3}{2}}$ >>>> when $f^{/} = 2$

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

مهم

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Chain rule:

In order to differentiate a function of a function, y = f(g(x)), that is to find $\frac{dy}{dx}$ we need to do two things:

1. Substitute u = g(x). This gives us

$$y = f(u)$$

Next, we need to use a formula that is known as the Chain Rule:

2. Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Key Point

Chain rule:
To differentiate $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example

Suppose we want to differentiate y = cosx^2.

Let u = x^2 so that y = cos u.

It follows immediately that

u = x^2 ……………….. $\frac{du}{dx} = 2x$

y = $\cos\ u$ …………… $\frac{dy}{du} = -\sin\ u$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin\ u. 2x = -2x. \sin\ u$

u = x^2 ⟶ $\frac{dy}{dx} = -2x \sin x^2$

Example

Suppose we want to differentiate $y = \cos^2 x = (\cos x)^2$.

Let $u = \cos x$ so that $y = u^2$

It follows that

$$\frac{du}{dx} = -\sin x \qquad \frac{dy}{du} = 2u$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2u \times -\sin x \\ &= -2\cos x \sin x \end{aligned}$$

Example

Suppose we wish to differentiate y = $(2x - 5)^{10}$.

Now it might be tempting to say ‘surely we could just multiply out the brackets’. To multiply out the brackets would take a long time and there are lots of opportunities for making mistakes. So let us treat this as a function of a function.

Let $u = 2x - 5$ so that $y = u^{10}$. It follows that

$$\frac{du}{dx} = 2 \qquad \frac{dy}{du} = 10u^9$$

Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 10u^9 \times 2$$
$$= 20(2x - 5)^9$$

هل من الممكن تغيير انواع المتغيرات في السؤال ؟ الجواب نعم

If y is a function of x , say $y=f(x)$, and x is a function of t, say $x=g(t)$ then y is a function of t :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

This formula is called chain rule.

Example: If $y = x^3 - x^2 + 5$ *and* $x = 2t^2 + t$, *find* $\frac{dy}{dt}$ *at* $t = 1$

Sol:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2 - 2x)(4t + 1)$$

$$at\ t = 1 \rightarrow x = (2)1^2 + 1 = 3$$

$$\frac{dy}{dt} = (3 * 3^2 - 2 * 3)(4 * 1 + 1) = 105$$

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Example : If $y = u^2 - 2u$ and $u = \sqrt{3x+1}$, find $\frac{dy}{dx}$

$$\frac{dy}{du} = 2u - 2 = 2(u-1) \quad \text{and} \quad \frac{du}{dx} = \frac{3}{2\sqrt{3x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2(u-1) \times \frac{3}{2\sqrt{3x+1}} = \frac{3(u-1)}{\sqrt{3x+1}} = \frac{3\left(\sqrt{3x+1}-1\right)}{\sqrt{3x+1}}$$

Example : If $y = t + \frac{1}{t}$ and $x = t - \frac{1}{t}$, find $\frac{dy}{dx}$

$$\frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2} \quad \text{and} \quad \frac{dx}{dt} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2-1}{t^2} \times \frac{t^2}{t^2+1} = \frac{t^2-1}{t^2+1}$$

Exercises

1. Find the derivative of each of the following:

a) $(3x-7)^{12}$ b) $\sin(5x+2)$ c) $\ln(2x-1)$ d) e^{2-3x}

e) $\sqrt{5x-3}$ f) $(6x+5)^{5/3}$ g) $\frac{1}{(3-x)^4}$ h) $\cos(1-4x)$

Answers

1. a) $36(3x-7)^{11}$ b) $5\cos(5x+2)$ c) $\frac{2}{2x-1}$ d) $-3e^{2-3x}$

e) $\frac{5}{2\sqrt{5x-3}}$ f) $10(6x+5)^{2/3}$ g) $\frac{4}{(3-x)^5}$ h) $4\sin(1-4x)$

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Implicit Differentiation:

Most of the functions we have dealt with so far have been described by an equation of the form $y=f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$y^2 - x^2 = 0, y^3 + 8x = 3, \qquad yx + 4y - 6 = 0$$

Definition:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.

2. Collect the terms with dy / dx on one side of the equation and solve for dy / dx.

Example: Find $\frac{dy}{dx}$ for the equation $y^2 + x^3 - 9xy = 0$

Sol:

$$2y\frac{dy}{dx} + 3x^2 - \left(9x\frac{dy}{dx} + 9y\right) = 0 \rightarrow 2y\frac{dy}{dx} + 3x^2 - 9x\frac{dy}{dx} - 9y = 0$$

$$2y\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \rightarrow \frac{dy}{dx}(2y - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y-3x^2}{2y-9x}$$

We can refer to $\frac{dy}{dx} = as\ a\ y'$ and $\frac{d^2y}{dx^2} = as\ a\ y''etc.$. وذلك لسهولة الحل

هل من الممكن اعادة المثال اعلاه باستخدام طريقة الاختصار أعلاه؟ حاول؟

Example: Use implicit differentiation to find $dy/\ dx$ for the equations

1- $y^2 = \frac{x-1}{x+1}$

2- $x cos\ (2x + 3\ ) = y \sin x$

Sol: 1- $y^2 = \frac{x-1}{x+1}$

$$2y\frac{dy}{dx} = \frac{(x+1).1-(x-1).1}{(x-1)^2} = \frac{x+1-x+1}{(x-1)^2} = \frac{2}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{2}{2y(x-1)^2} = \frac{1}{y(x-1)^2}$$

2- $x cos\ (2x + 3) = y \sin x$

Sol:

$$-x \sin(2x+3)\,.\,2 + \cos\ (2x+3)\,.\,1 = y\cos x + \frac{dy}{dx}\sin x$$

$$-2x\sin(2x+3) + \cos(2x+3) = y\cos x + \frac{dy}{dx}\sin x$$

$$-2x\sin(2x+3) + \cos(2x+3) - y\cos x = \frac{dy}{dx}\sin x$$

$$\frac{dy}{dx} = \frac{-2x\sin(2x+3) + \cos(2x+3) - y\cos x}{\sin x}$$

Example: find dy/dx for the equation $y^2 = X^2 + \sin xy$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(x\frac{dy}{dx} + y\right)$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx} + y\right) = 2x$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$

$$\big(2y - x(\cos xy)\big)\frac{dy}{dx} = 2x + y(\cos xy)$$

$$\frac{dy}{dx} = \frac{2x + y(\cos xy)}{2y - x(\cos xy)}$$

Home Work

find d^2y/dx^2 for the equation $2x^3 - 3y^2 = 8$

Find the derivative of the following function:

$$x^2y^2 = x^2 + y^2$$