Al-Mustaqbal University
College of Engineering Technology
Department of Cyber Security Techniques Engineering
Class: 1st
Subject: Math II
Lecturer: Dr. Hussein Ali Ameen
2nd term – Vector Product

Operations on vectors

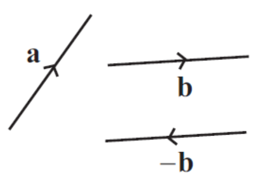
One of the things we can do with vectors is to add them together. We shall start by adding two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on.

Key Point

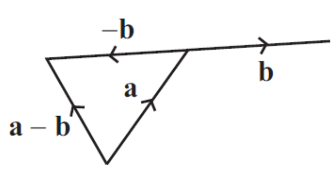
We can add two vectors $\mathbf{a}$ and $\mathbf{b}$ by making $\mathbf{b}$ start where $\mathbf{a}$ finishes, and completing the triangle. Alternatively, we can make $\mathbf{a}$ and $\mathbf{b}$ start at the same place, and take the diagonal of the parallelogram.

Subtracting two vectors

What is $\mathbf{a} - \mathbf{b}$? We think of this as $\mathbf{a} + (-\mathbf{b})$, and then we ask what $-\mathbf{b}$ might mean. This will be a vector equal in magnitude to $\mathbf{b}$, but in the reverse direction.



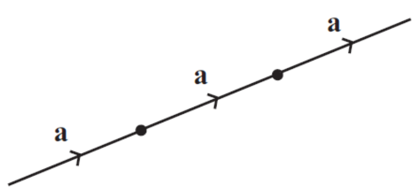
Now we can subtract two vectors. Subtracting $\mathbf{b}$ from $\mathbf{a}$ will be the same as *adding* $-\mathbf{b}$ to $\mathbf{a}$.



Key Point

$\mathbf{a} - \mathbf{b}$ means $\mathbf{a} + (-\mathbf{b})$

What happens when you add a vector to itself, perhaps several times? We write, for example, $\mathbf{a} + \mathbf{a} + \mathbf{a} = 3\mathbf{a}$.



In the same way, we would write

$$n\mathbf{a} = \underbrace{\mathbf{a} + \ldots + \mathbf{a}}_{n \text{ copies}}.$$

Key Point

A vector $n\mathbf{a}$ is in the same direction as the vector $\mathbf{a}$, but n times as long.

What is a dot product (Inner product)?

- Dot product or Inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is represented as:
  - $\boldsymbol{a} \cdot \boldsymbol{b} = s$
- Dot product of two vectors results in a scalar.
- Multiply the corresponding components of the two vectors
- The dot product equals to the result of addition of all the multiplied components
- $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3.$
- Example:
  - $\begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 - 1 = 1$
  - $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ \sqrt{2} \end{bmatrix} = 3\times 2 + (-1)\times 1 + 0\times\sqrt{2} = 5$

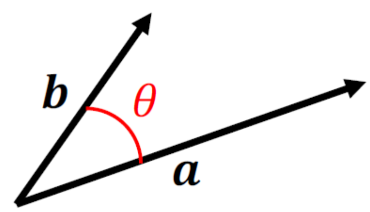
How to dot two vectors

- Dot product can be computed only between vectors of same dimension.
- Dot product is commutative
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  - $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3\times 2 + (-1)\times 1 + 0\times\sqrt{2} = 5$

$$|A| = \sqrt{a^2 + b^2}$$

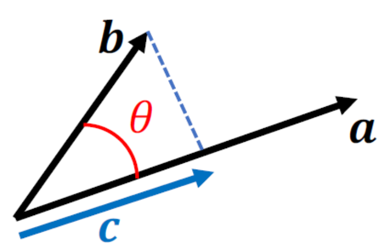
Geometric interpretation of a dot product

- The result of a dot product of vectors is a scalar, and cannot be depicted as a vector.
- However, this scalar value is proportional to the cosine of the angle between the vectors.
- So dot product can be computed in two different ways. One as the sum of the product of the corresponding components as mentioned earlier, and the other as
  $$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \, \cos\theta$$
- Both computation methods will yield the same result.
- Example:
  - $\boldsymbol{a} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times 2 \times \cos 60^{\circ} = 2$
  - By earlier approach, $\boldsymbol{a} \cdot \boldsymbol{b} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ = 0+2=2



Geometric interpretation of a dot product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \, \cos\theta$
- What if $\boldsymbol{a}$ is a unit vector ($\|\boldsymbol{a}\|$=1)
  - $\boldsymbol{a} \cdot \boldsymbol{b}$ would be the length of the perpendicular projection of $\boldsymbol{b}$ on $\boldsymbol{a}$
- Vector $\boldsymbol{c}$ is the image of $\boldsymbol{b}$ on $\boldsymbol{a}$
- Direction of $\boldsymbol{c}$ is the same as $\boldsymbol{a}$
- Magnitude of $\boldsymbol{c}$ is
  - $\|\boldsymbol{c}\| = \|\boldsymbol{b}\| \, \cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|}$



Cross product of parallel vectors

- From the geometrical representation of cross product it is inferred that the cross product of parallel vectors is a zero vector

$$\text{a x b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin 0^{\circ} = 0$$

- i.e. cross product of a vector with itself is zero vector

$$\text{a x a} = \|\boldsymbol{a}\|\|\boldsymbol{a}\| \sin 0^{\circ} = 0$$

Dot Product and Cross Product (Cos, Sin) Related to cosine and sine wave

Discussion

Cross product of orthogonal vectors

- From the geometric representation of cross product, it is inferred that the cross product of two orthogonal vector is the product of their magnitude.

$$\text{a x b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin 90^{\circ} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|$$

Application of Cross product

- To find the area of a triangle.

$$\text{Area of triangle} = \frac{1}{2} h \, |\underline{B}|$$
$$= \frac{1}{2} |\underline{A}| \sin\theta \, |\underline{B}|$$
$$= \frac{1}{2} | \underline{A} \times \underline{B} |$$