



Definition

The derivative of a function $f(x)$ with respect to the variable x is the function $f'(x)$ whose value at x is:

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Provided this limit exists.

Differentiation Rules:

1-Derivative of a constant function

If $y=f(x)=c$ where c is a constant then

$$\frac{dy}{dx} = f'(x) = 0$$

Example1: $f(x)=1$ then $\frac{dy}{dx} = f' = 0$

Example2: $f(x)=25$ then $\frac{dy}{dx} = f' = 0$

Example3: $f(x)=-1$ then $\frac{dy}{dx} = f' = 0$

Example4: $f(x)=1000$ then $\frac{dy}{dx} = f' = 0$



2-Power rule for positive integers:

If n is a positive integer, and $y = f(x) = x^n$;then

$$\dot{f}(x) = \frac{dy}{dx} = n x^{n-1}$$

Example: $y = f(x) = x^3$ then $\dot{f}(x) = \frac{dy}{dx} = 3 x^2$

Example: $y = f(x) = x^8$ then $\dot{f}(x) = \frac{dy}{dx} = 8 x^7$

$$f(x) = x^2$$

$$f'(x) = nx^{n-1}$$

$$f(x) = x^n \text{ مشتقة المتغير } - ٢$$

3-Derivative constant multiple rule:

If u is a differentiable function of x $f(x) = c u(x)$, and c is a constant, then

$$\frac{dy}{dx} = \dot{f}(x) = c \dot{u}(x)$$

Example: $f(x) = 5x^7$ then $\frac{dy}{dx} = 5 * 7 x^6 = 35 x^6$

Example: $f(x) = 2x^4$ then $\frac{dy}{dx} = 2 * 4 x^3 = 8 x^3$



1) $f(x) = x$	$\Rightarrow$	$f'(x) = 1$
2) $f(x) = x^2$	$\Rightarrow$	$f'(x) = 2x$
3) $f(x) = x^3$	$\Rightarrow$	$f'(x) = 3x^2$
4) $f(x) = x^4$	$\Rightarrow$	$f'(x) = 4x^3$
5) $f(x) = x^5$	$\Rightarrow$	$f'(x) = 5x^4$
6) $f(x) = 2x$	$\Rightarrow$	$f'(x) = 2$
7) $f(x) = 3x^2$	$\Rightarrow$	$f'(x) = 6x$
8) $f(x) = 5x^3$	$\Rightarrow$	$f'(x) = 15x^2$
9) $f(x) = x^{-2}$	$\Rightarrow$	$f'(x) = -2x^{-3}$
10) $f(x) = x^{-3}$	$\Rightarrow$	$f'(x) = -3x^{-4}$
11) $f(x) = x^{-4}$	$\Rightarrow$	$f'(x) = -4x^{-5}$
12) $f(x) = 3x^{-5}$	$\Rightarrow$	$f'(x) = -15x^{-6}$

مشتقة الجذور

1) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$	$\Rightarrow$	$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
2) $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$	$\Rightarrow$	$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$
3) $f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$	$\Rightarrow$	$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$
4) $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$	$\Rightarrow$	$f'(x) = \frac{1}{5}x^{-\frac{4}{5}}$
5) $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$	$\Rightarrow$	$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$
6) $f(x) = \sqrt[5]{x^3} = x^{\frac{3}{5}}$	$\Rightarrow$	$f'(x) = \frac{3}{5}x^{-\frac{2}{5}}$
7) $f(x) = \sqrt[3]{x^{-2}} = x^{-\frac{2}{3}}$	$\Rightarrow$	$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}}$



Note

Derivative of polynomial functions: Differentiate each term of the polynomial separately and keep the plus/minus signs

$$f(x) = h(x) \mp g(x) \implies f'(x) = h'(x) \mp g'(x)$$

$$1) f(x) = 3x^5 + 7x \implies f'(x) = 15x^4 + 7$$

$$2) f(x) = 3x^4 - 4x^2 + 6 \implies f'(x) = 12x^3 - 8x$$

$$3) f(x) = 2x^2 + \frac{1}{2}x \implies f'(x) = 4x + \frac{1}{2}$$

$$4) f(x) = \frac{1}{2}x^2 - \frac{4}{3}x^3 + 9 \implies f'(x) = x - 4x^2$$

$$5) f(x) = \frac{1}{5}x^{-2} - \frac{2}{7}x^{-3} + 9 \implies f'(x) = \frac{-2}{5}x^{-3} + \frac{6}{7}x^{-4}$$

4-Derivative Product rule:

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

٤- مشتقة حاصل ضرب دالتين = الاولى في مشتقة الثانية + الثانية في مشتقة الاولى

$$f(x) = g(x) . h(x) \implies f'(x) = g(x) . h'(x) + h(x) . g'(x)$$

Example: Find the derivative of $y = (x^2) . (x^3)$

Sol:

$$\frac{dy}{dx} = (x^2) . 3x^2 + (x^3) . (2x)$$



Example: Find the derivative of $y = (x^2 + 8x)(x^3 - 1)$

Sol:

$$\frac{dy}{dx} = (x^2 + 8x) \cdot 3x^2 + (x^3 - 1) \cdot (2x + 8)$$

Examples:

Find the derivative of the $f(x) = (x^4 - x^2 + 1)(5x^6 - 3x)$

$$f(x) = (x^4 - x^2 + 1)(5x^6 - 3x)$$

$$f'(x) = (x^4 - x^2 + 1)(30x^5 - 3) + (5x^6 - 3x)(4x^3 - 2x)$$

$$x=2 \Rightarrow f(x) = (4 - x)(x^2 + 3)$$

$$f(x) = (4 - x)(x^2 + 3)$$

$$f'(x) = (4 - x)(2x) + (x^2 + 3)(-1)$$

$$f'(2) = (4 - 2)(2(2)) + (2^2 + 3)(-1) = 2(4) + (7)(-1) = 8 - 7 = 1$$

5-Derivative quotient rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient $\frac{u}{v}$ is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$



٥- مشتقة حاصل قسمة دالتين = $\frac{\text{المقام في مشتقة البسط} - \text{البسط في مشتقة المقام}}{\text{المقام}^2}$

$$f(x) = \frac{g(x)}{h(x)} \longrightarrow f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{(h(x))^2}$$

Example: Find the derivative of $y = \frac{x^2}{x^3}$

Sol:

$$\frac{dy}{dx} = \frac{(x^3) \cdot (2x) - (x^2) \cdot (3x^2)}{(x^3)^2} = \frac{2x^4 - 3x^4}{x^6}$$

Example: Find the derivative of $y = \frac{t-t^2}{t+4}$

Sol:

$$\frac{d}{dx} \left(\frac{t-t^2}{t+4} \right) = \frac{(t+4) \cdot (1-2t) - (t-t^2) \cdot 1}{(t+4)^2} = \frac{(t+4)(1-2t) - (t-t^2)}{(t+4)^2}$$

Example: Find the derivative of $f(x) = \frac{x^3+1}{x^4+1}$ at $x=1$

$$f'(x) = \frac{(x^4+1)(3x^2) - (x^3+1)(4x^3)}{(x^4+1)^2}$$

$$f'(1) = \frac{(1^4+1)(3 \times 1^2) - (1^3+1)(4 \times 1^3)}{(1^4+1)^2} = \frac{2 \times 3 - 2 \times 4}{2^2} = \frac{6-8}{4} = \frac{-2}{4} = \frac{-1}{2}$$

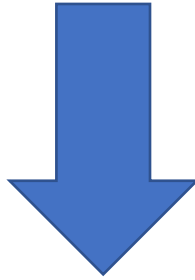


Example: Find the derivative of $f(x) = \frac{4-5x}{x^4+x+1}$ at $x = -1$

$$f'(x) = \frac{(x^2+x+1)(-5) - (4-5x)(2x+1)}{(x^2+x+1)^2}$$

$$f'(-1) = \frac{((-1)^2+(-1)+1)(-5) - (4-5(-1))(2(-1)+1)}{((-1)^2+(-1)+1)^2}$$

$$f'(-1) = \frac{(1-1+1)(-5) - (4+5)(-2+1)}{(1-1+1)^2} = \frac{-5-(9)(-1)}{1} = -5 + 9 = 4$$



مشتق القوس مرفوعة الى أس = الاس في القوس مرفوع الى الأس - 1 في مشتقة داخل القوس

$$f(x) = [h(x)]^n \implies f'(x) = n[h(x)]^{n-1} (h'(x))$$

$$f(x) = (1-x)^3$$

$$f'(x) = 3(1-x)^2 (-1) = -3(1-x)^2$$

$$f(x) = (x^3 + x^2 + x + 1)^5$$

$$f'(x) = 5(x^3 + x^2 + x + 1)^4 (3x^2 + 2x + 1)$$



Second and Higher- Order Derivatives:

If $f(x)$ is a given function then

$\frac{dy}{dx} = \dot{f}(x)$ is first derivative of y .

$\frac{d^2y}{dx^2} = \ddot{f}(x)$ is second derivative of y .

$\frac{d^3y}{dx^3} = \dddot{f}(x)$ is third derivative of y . And so on...

Then, in general:

$$\frac{d^ny}{dx^n} = f^n(x) = y^n$$

Example: If $y = (x^2 + 2x + 3)^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Sol:

$$\frac{dy}{dx} = \dot{y} = 2 \cdot (x^2 + 2x + 3)(2x + 2)$$

$$\frac{d^2y}{dx^2} = 2 [(x^2 + 2x + 3) \cdot 2 + (2x + 2) \cdot (2x + 2)]$$

$$\frac{d^2y}{dx^2} = 4(x^2 + 2x + 3) + 2(2x + 2)^2$$



Example:

If $y = (x^2 + 2x + 3)^2$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$

Sol:

$$\frac{dy}{dx} = 6x^2 - 8x + 6$$

$$\frac{d^2y}{dx^2} = 12x$$

$$\frac{d^3y}{dx^3} = 12$$

$$\frac{d^4y}{dx^4} = 0$$

Example:

Find the value of $f'(x)$, and $f''(x)$ if $f(x) = (x^2 - 3)^4$

$$f'(x) = 4(x^2 - 3)^3 (2x) = 8x(x^2 - 3)^3$$

$$f'(2) = 8(2)(2^2 - 3)^3 = 16(1)^3 = 16$$

$$f''(x) = 8x(3)(x^2 - 3)^2 (2x) + (x^2 - 3)^3 (8)$$

$$f''(x) = 48x^2(x^2 - 3)^2 + 8(x^2 - 3)^3$$

$$f''(2) = 48(2)^2(2^2 - 3)^2 + 8(2^2 - 3)^3 = 192 + 8 = 200$$



Note

ملاحظة / مشتقة الجذر التربيعي = مشتقة داخل الجذر على ٢ في الجذر

$$f(x) = \sqrt{\quad} \longrightarrow f'(x) = \frac{\text{مشتقة داخل الجذر}}{2\sqrt{\quad}}$$

$$f'(x) \nrightarrow f(x) = \sqrt{x^3 + x^2 - 5}$$

$$f'(x) = \frac{3x^2 + 2x}{2\sqrt{x^3 + x^2 - 5}}$$

$$f'(x) \nrightarrow f(x) = \sqrt{x^2 - 2x + 1}$$

$$f'(x) = \frac{2x - 2}{2\sqrt{x^2 - 2x + 1}}$$

$$y'', y' \nrightarrow y = x^4 + 5x^3 + 3$$

$$y' = 4x^3 + 15x^2$$

$$y'' = 12x^2 + 30x$$

Find the value
of f(x)

$$f''(-1), f''(x), f'(x) \nrightarrow f(x) = 2x^3 + 4 + \frac{3}{x}$$

$$f(x) = 2x^3 + 4 + 3x^{-1}$$

$$f'(x) = 6x^2 - 3x^{-2} \implies f'(x) = 6x^2 - \frac{3}{x^2}$$

$$f''(x) = 12x + 6x^{-3} \implies f''(x) = 12x + \frac{6}{x^3}$$

$$f''(-1) = 12(-1) + \frac{6}{(-1)^3} = -12 - 6 = -18$$



Find the value of $f(x)$ $x = 0$ at $f(x) = \frac{1}{\sqrt{2x+1}}$

$$f(x) = (2x + 1)^{-\frac{1}{2}}$$

$$f'(x) = \frac{-1}{2}(2x + 1)^{-\frac{1}{2}}(2) = -\sqrt{2x + 1}$$

$$f'(0) = -\sqrt{2(0) + 1} = -\sqrt{1} = -1$$

Home Work

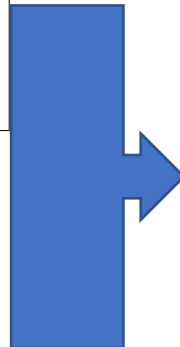
EX/ Find f' of functions

(1) $f(x) = \left(\frac{x}{x+1}\right)^4$ >>>>> when $x = 1$

(2) $f(x) = x + \frac{3}{x^2+1}$ >>>>> when $x = -1$

(3) $f(x) = (x^3 + 3x^2 - 3)^{\frac{3}{2}}$ >>>> when $f' = 2$

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$





Chain rule:

In order to differentiate a function of a function, $y = f(g(x))$, that is to find $\frac{dy}{dx}$ we need to do two things:

1. Substitute $u = g(x)$. This gives us

$$y = f(u)$$

Next, we need to use a formula that is known as the Chain Rule:

2. Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$



Key Point

Chain rule:

To differentiate $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example

Suppose we want to differentiate $y = \cos x^2$.

Let $u = x^2$ so that $y = \cos u$.

It follows immediately that



$$u = x^2 \dots\dots\dots \frac{du}{dx} = 2x$$

$$y = \cos u \dots\dots\dots \frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin u \cdot 2x = -2x \cdot \sin u$$

$$u = x^2 \longrightarrow \frac{dy}{dx} = -2x \sin x^2$$

Example

Suppose we want to differentiate $y = \cos^2 x = (\cos x)^2$.

Let $u = \cos x$ so that $y = u^2$

It follows that

$$\frac{du}{dx} = -\sin x \qquad \frac{dy}{du} = 2u$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2u \times -\sin x \\ &= -2 \cos x \sin x \end{aligned}$$

Example

Suppose we wish to differentiate $y = (2x - 5)^{10}$. 

Now it might be tempting to say ‘surely we could just multiply out the brackets’. To multiply out the brackets would take a long time and there are lots of opportunities for making mistakes. So let us treat this as a function of a function.



Let $u = 2x - 5$ so that $y = u^{10}$. It follows that

$$\frac{du}{dx} = 2 \qquad \frac{dy}{du} = 10u^9$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 10u^9 \times 2 \\ &= 20(2x - 5)^9 \end{aligned}$$

هل من الممكن تغيير انواع المتغيرات في السؤال ؟ الجواب نعم

If y is a function of x , say $y=f(x)$, and x is a function of t , say $x=g(t)$ then y is a function of t :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

This formula is called chain rule.

Example: If $y = x^3 - x^2 + 5$ and $x = 2t^2 + t$, find $\frac{dy}{dt}$ at $t = 1$.

Sol:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2 - 2x)(4t + 1)$$

$$\text{at } t = 1 \rightarrow x = (2)1^2 + 1 = 3$$

$$\frac{dy}{dt} = (3 * 3^2 - 2 * 3)(4 * 1 + 1) = 105$$



Example : If $y = u^2 - 2u$ and $u = \sqrt{3x+1}$, find $\frac{dy}{dx}$

$$\frac{dy}{du} = 2u - 2 = 2(u - 1) \quad \text{and} \quad \frac{du}{dx} = \frac{3}{2\sqrt{3x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2(u - 1) \times \frac{3}{2\sqrt{3x+1}} = \frac{3(u - 1)}{\sqrt{3x+1}} = \frac{3(\sqrt{3x+1} - 1)}{\sqrt{3x+1}}$$

Example : If $y = t + \frac{1}{t}$ and $x = t - \frac{1}{t}$, find $\frac{dy}{dx}$

$$\frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2} \quad \text{and} \quad \frac{dx}{dt} = 1 + \frac{1}{t^2} = \frac{t^2 + 1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2 - 1}{t^2} \times \frac{t^2}{t^2 + 1} = \frac{t^2 - 1}{t^2 + 1}$$

Exercises

1. Find the derivative of each of the following:

- a) $(3x - 7)^{12}$ b) $\sin(5x + 2)$ c) $\ln(2x - 1)$ d) e^{2-3x}
 e) $\sqrt{5x - 3}$ f) $(6x + 5)^{5/3}$ g) $\frac{1}{(3 - x)^4}$ h) $\cos(1 - 4x)$

Answers

1. a) $36(3x - 7)^{11}$ b) $5 \cos(5x + 2)$ c) $\frac{2}{2x - 1}$ d) $-3e^{2-3x}$
 e) $\frac{5}{2\sqrt{5x - 3}}$ f) $10(6x + 5)^{2/3}$ g) $\frac{4}{(3 - x)^5}$ h) $4 \sin(1 - 4x)$



Implicit Differentiation:

Most of the functions we have dealt with so far have been described by an equation of the form $y=f(x)$ that expresses y explicitly in terms of the variable x . We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$y^2 - x^2 = 0, y^3 + 8x = 3, \quad yx + 4y - 6 = 0$$

Definition:

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx .

Example: Find $\frac{dy}{dx}$ for the equation $y^2 + x^3 - 9xy = 0$

Sol:

$$2y \frac{dy}{dx} + 3x^2 - \left(9x \frac{dy}{dx} + 9y \right) = 0 \rightarrow 2y \frac{dy}{dx} + 3x^2 - 9x \frac{dy}{dx} - 9y = 0$$

$$2y \frac{dy}{dx} - 9x \frac{dy}{dx} = 9y - 3x^2 \rightarrow \frac{dy}{dx} (2y - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{2y - 9x}$$

We can refer to $\frac{dy}{dx} = \text{as a } y'$ and $\frac{d^2y}{dx^2} = \text{as a } y'' \text{ etc..}$ وذلك لسهولة الحل

هل من الممكن إعادة المثال أعلاه باستخدام طريقة الاختصار أعلاه؟ حاول!



Example: Use implicit differentiation to find dy/dx for the equations

$$1- y^2 = \frac{x-1}{x+1}$$

$$2- x \cos (2x + 3) = y \sin x$$

Sol: $1- y^2 = \frac{x-1}{x+1}$

$$2y \frac{dy}{dx} = \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x+1)^2} = \frac{x+1 - x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{2}{2y(x+1)^2} = \frac{1}{y(x+1)^2}$$

$$2- x \cos (2x + 3) = y \sin x$$

Sol:

$$-x \sin(2x + 3) \cdot 2 + \cos (2x + 3) \cdot 1 = y \cos x + \frac{dy}{dx} \sin x$$

$$-2x \sin(2x + 3) + \cos(2x + 3) = y \cos x + \frac{dy}{dx} \sin x$$

$$-2x \sin(2x + 3) + \cos(2x + 3) - y \cos x = \frac{dy}{dx} \sin x$$

$$\frac{dy}{dx} = \frac{-2x \sin(2x + 3) + \cos(2x + 3) - y \cos x}{\sin x}$$



Example: find dy/dx for the equation $y^2 = x^2 + \sin xy$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$2y \frac{dy}{dx} = 2x + (\cos xy) \left(x \frac{dy}{dx} + y \right)$$

$$2y \frac{dy}{dx} - (\cos xy) \left(x \frac{dy}{dx} + y \right) = 2x$$

$$2y \frac{dy}{dx} - (\cos xy) \left(x \frac{dy}{dx} \right) = 2x + (\cos xy)y$$

$$(2y - x(\cos xy)) \frac{dy}{dx} = 2x + y(\cos xy)$$

$$\frac{dy}{dx} = \frac{2x + y(\cos xy)}{2y - x(\cos xy)}$$

Home Work

find d^2y/dx^2 for the equation $2x^3 - 3y^2 = 8$

Find the derivative of the following function:

$$x^2y^2 = x^2 + y^2$$