كلية التقنيات الهندسية

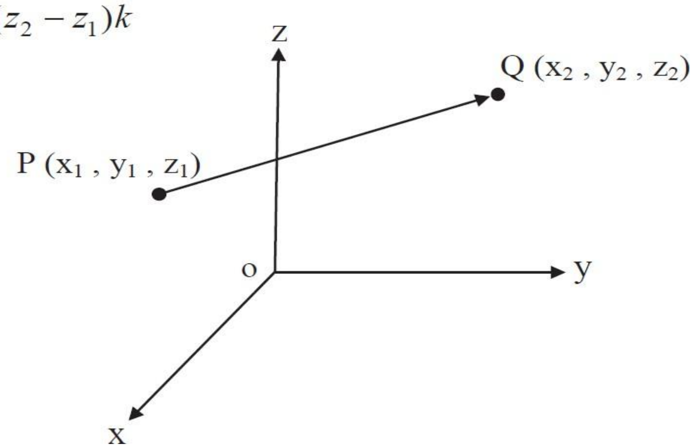
هندسة تقنيات الذكاء الاصطناعي

Lecture 2 vectors (dot product, cross product, orthogonal)

Vector in space:

The vector between two points **P** and **Q**

$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$



Algebra of vectors:

Let $V_1 = a_1 i + b_1 j + c_1 k$

$V_2 = a_2 i + b_2 j + c_2 k$

Addition:

$$V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j + (c_1 + c_2)k$$

Subtraction:

$$V_1 - V_2 = (a_1 - a_2)i + (b_1 - b_2)j + (c_1 - c_2)k$$

***The magnitude or length of the vector*:**

$V = ai + bj + ck$

$$|V| = \sqrt{a^2 + b^2 + c^2}$$

***Direction of vectors* V:** $\quad V = \dfrac{V}{|V|}$

***Example*:** find the length of vector $A = i - 2j + 3k$

***Solution*:**

$$|A| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$
$$= \sqrt{1+4+9} = \sqrt{14}$$

***Example*:** A force of 6N is applied in the direction of the vector $V = 2i + 2j - k$, express the force as a product of its magnitude and direction

***Solution*:**

The force vector has magnitude 6 and direction $\dfrac{V}{|V|}$, so:

$$F = 6\frac{V}{|V|} = 6\frac{2i + 2j - k}{\sqrt{(2)^2 + (2)^2 + (-1)^2}} = 6\frac{2i + 2j - k}{\sqrt{9}}$$

$$= 6\frac{2i + 2j - k}{3}$$

***H.W*:**

1. Let the vector $u = -i + 3j + k$, find $\left|\frac{1}{2}u\right|$

2. find the length and direction of the vector $v = 4i + 3j + 2k$

3. find the length of the vector with initial point P(-3,4,1) and terminal point Q(-5,2,2)

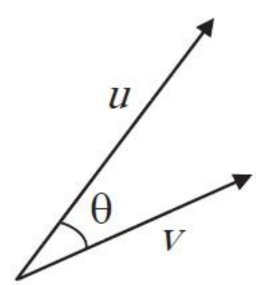
Dot Product

***The dot product*:**

Dot product also called scalar products because the resulting products are numbers and not vectors. To calculate ***u.v*** from the component of ***u*** and ***v*** we let:

$$u = a_1 i + b_1 j + c_1 k$$

$$v = a_2 i + b_2 j + c_2 k$$



$$u.v = |u||v|\cos\theta$$

Where θ is the angle between ***u*** and ***v***

$$\cos\theta = \frac{u.v}{|u||v|}$$

$$u.v = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Properties of dot product:

if ***u*** , ***v***, and ***w*** are any vectors and ***c*** is a scalar:

1. $u.v = v.u$

2. $(cu).v = u.(cv) = c(u.v)$

3. $u.(v+w) = u.v + u.w$

4. $u.u = |u|^2$

5. $0.u = 0$

Example: find the angle θ between $A = i - 2j - 2k$ and $B = 6i + 3j + 2k$

Solution:

$$A.B = (1)(6) + (-2)(3) + (-2)(2)$$

$$= 6 - 6 - 4 = -4$$

$$|A| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|B| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{-4}{21} \Longrightarrow \theta = \cos^{-1}\frac{-4}{21} = 100.79^o \approx 101^o$$

Example: find the angle between the vectors $u = 2i + j$, $v = i + 2j - k$

Solution:

$$u.v = (2)(1) + (1)(2) + (0)(-1)$$

$$= 2 + 2 - 0 = 4$$

$$|u| = \sqrt{(2)^2 + (1)^2 + (0)^2} = \sqrt{4+1+0} = \sqrt{5}$$

$$|v| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{u.v}{|u||v|} = \frac{4}{\sqrt{5}\sqrt{6}} = \frac{4}{\sqrt{30}}$$

$$\theta = \cos{-1}\frac{4}{\sqrt{30}} = 43.09^o \approx 43^o$$

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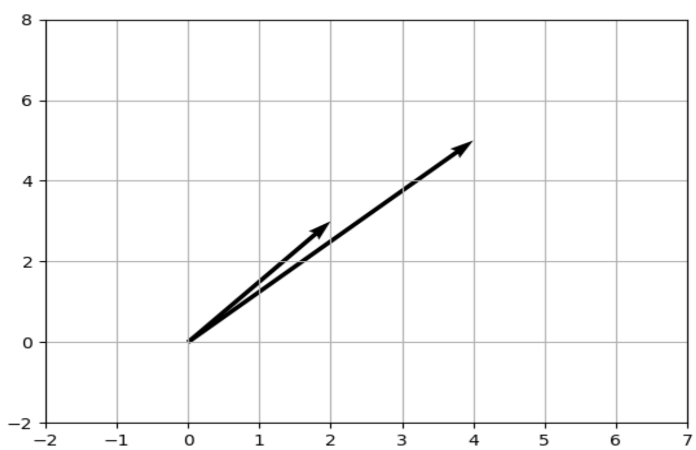
Example

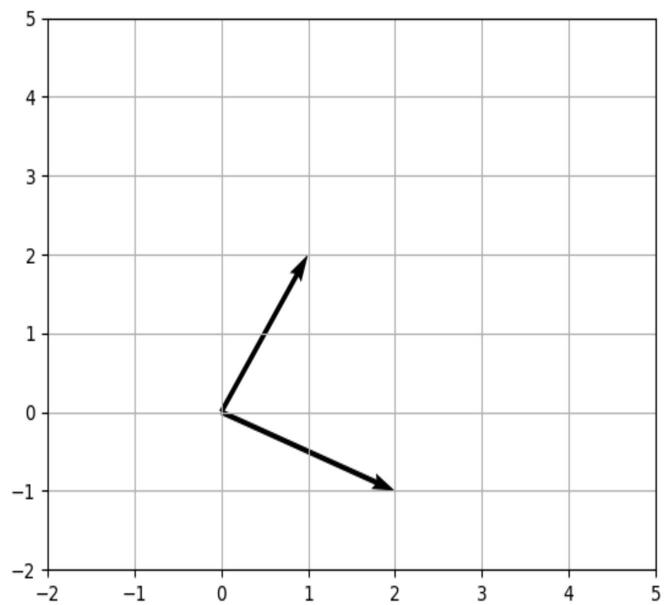
The vector A=(2,3), and vector B=(4,5)
find A·B = ?



Orthogonal Vectors

A=(2,-1), B=(1,2)

A·B = 0 → vectors are perpendicular (90°)



Applications

- Physics: Work = Force · Displacement
- AI: Similarity between vectors
- Engineering: Projection of forces.

Practice Questions

1. (3,2)·(1,4)
2. (5,-1)·(2,3)
3. Are (1,2) and (-2,1) orthogonal?

Cross product:

Two vector ***u*** and ***v*** in space if ***u*** and ***v*** are not parallel, they determine a plane, we select a unit vector ***n*** perpendicular to the plane by the right-hand rule. This means that we choose ***n*** to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from ***u*** to ***v***

Properties of the cross product

if ***u***, ***v***, and ***w*** are any vectors and ***r***, ***s*** is are scalars, then:

1. $(ru)\times(sv)=(rs)(u\times v)$

2. $u\times(v+w)=u\times v+u\times w$

3. $v\times u=-(u\times v)$

4. $(v+w)\times u=v\times u+w\times u$

5. $0\times u=0$

6. $u\times(v\times w)=(u.w)v-(u.v)w$

When we apply the definition to calculate the pair wise cross products of ***i***, ***j***, ***k*** we fined:

$$i\times j=-(j\times i)=k$$
$$j\times k=-(k\times j)=i$$
$$k\times i=-(i\times k)=j$$
$$i\times i=j\times j=k\times k=0$$

Example: find $u \times v$ and $v \times u$ if $u = 2i + j + k$ and $v = -4i + 3j + k$

Solution:

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= ((1)(1) - (1)(3))i - ((2)(1) - (1)(-4))j + ((2)(3) - (1)(-4))k$$

$$= (1-3)i - (2+4)j + (6+4)k$$

$$= -2i - 6j + 10k$$

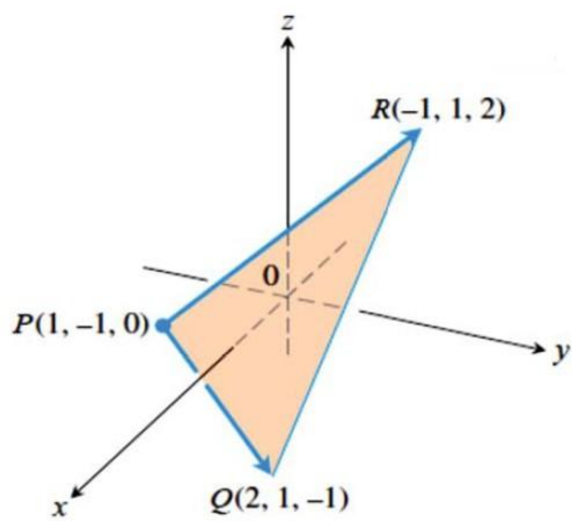
$$v \times u = -(u \times v) = 2i + 6j - 10k$$

Example: find the area of the triangle with vertices p(1,-1,0) , Q(2,1,-1) and R(-1,1,2)

Solution:

The area of the triangle is

$(\frac{1}{2})\ |\overrightarrow{PQ} \times \overrightarrow{PR}|$. in term of components



$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= ((2)(2) - (-1)(2))i - ((1)(2) - (-1)(-2))j + ((1)(2) - (2)(-2))k$$

$$= (4+2)i - (2-2)j + (2+4)k$$

$$= 6i + 6k$$

Hence, the triangle's area is:

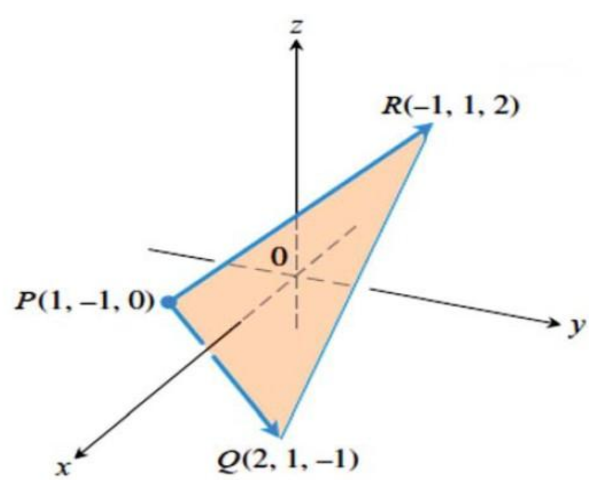
$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |6i + 6k| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$$

The triangle's area is half of this $= 3\sqrt{2}$

Example: find a unit vector perpendicular to the plane of p(1,-1,0) , Q(2,1,-1) and R(-1,1,2)

Solution:

Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the Plane, its direction n is a unit vector Perpendicular to the plane



$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}$$

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= 6i + 6k$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6i + 6k| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$$

$$n = \frac{6i + 6k}{6\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k$$

H.W: Triangle with points $P(1,-1,2)$, $Q(2,0,-1)$ and $R(0,2,1)$, find:

1. Area of the triangle.

2. a unit vector perpendicular to the plane PQR.