Al-Mustaqbal University
Department of Artificial Intelligence Techniques Engineering
Class: 1^{st}
Subject: Math
Lecturer: Dr. Hussein Ali Ameen
1^{st} term – Lecture: 3+4 Inequalities, Functions and Limits

Definition:

If a and b are real number, then one of the following is true

$\boldsymbol{a > b}$, $\boldsymbol{a < b}$, $\boldsymbol{a = b}$

If $\boldsymbol{a > b}$ **then** $\boldsymbol{-a < b \; ; \; -2 < -1}$

If $\boldsymbol{a > b}$ **then** $\boldsymbol{\frac{1}{a} < \frac{1}{b}} \Longrightarrow \boldsymbol{\frac{1}{5} < \frac{1}{4}}$

Intervals:

Definition: An interval is a set of numbers x having one of the following

1- Open interval: $a < x < b \equiv (a, b)$
2- Closed interval: $a \leq x \leq b \equiv [a, b]$
3- Half open from the left or half close from the right $a < x \leq b \equiv (a, b]$
4- Half close from the left or half open from the right $a \leq x < b \equiv [a, b)$

Notes:

1- $a < x < \infty \equiv a < x \equiv (a, \infty)$
2- $a \leq x < \infty \equiv a \leq x \equiv [a, \infty)$
3- $\infty < x < a \equiv x < a \equiv (-\infty, a)$
4- $\infty < x \leq a \equiv x \leq a \equiv (-\infty, a]$

Discussion

hussein_awadh@uomus.edu.iq

Absolute Value:

Definition: The absolute value of real number x is defined as:

$$|x| = \begin{cases} x & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Properties of absolute value:

$1 - |x| = a\ if\ and\ only\ if\ x = \pm a$

2- $|x| = |-x|$, a number and its additive inverse or negative have the same absolute value.

$3 - |x.y| = |x|.|y|\ and\ \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

$4 - |-x| = |x|, |a| = \sqrt{a^2}\quad where\ a = scaler.$

$5 - |x \pm y| \leq |x| \pm |y|$

$6 - |x| < a$ this means $-a < x < a$

$7 - |x| \geq a$ this means $-a \leq x \leq a$

$8 - |x| > a$ this means $x < -a\ \ or\ \ x > a$

$9 - |x| \geq a$ this means $x \leq -a\ \ or\ x \geq a$

hussein_awadh@uomus.edu.iq

Example: Find the absolute value of the following:

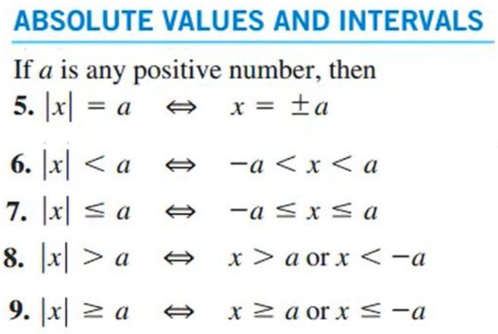
1) $\left|\frac{2x+1}{4}\right| \leq 6$ 2) $|5x-2| \geq 1$ 3) $|2x-3| \leq 1$

Solution:

1) $\left|\frac{2x+1}{4}\right| \leq 6 \rightarrow \left[-6 \leq \frac{2x+1}{4} \leq 6\right] * 4 \rightarrow -24 \leq 2x+1 \leq 24$

$$-1-24 \leq 2x+1-1 \leq 24-1 \quad \rightarrow \quad [-25 \leq 2x \leq 23] \div 2$$

$$-\frac{25}{2} \leq x \leq \frac{23}{2}$$



ABSOLUTE VALUES AND INTERVALS

If a is any positive number, then

5. $|x| = a \iff x = \pm a$

6. $|x| < a \iff -a < x < a$

7. $|x| \leq a \iff -a \leq x \leq a$

8. $|x| > a \iff x > a$ or $x < -a$

9. $|x| \geq a \iff x \geq a$ or $x \leq -a$

2) $|5x-2| \geq 1$

$5x-2 \geq 1$ or $5x-2 \leq -1$

$5x-2+2 \geq 1+2$ or $5x-2+2 \leq -1+2$

$5x \geq 3$ or $5x \leq 1$

$\frac{5x}{5} \geq \frac{3}{5}$ or $\frac{5x}{5} \leq \frac{1}{5}$

$x \geq \frac{3}{5}$ or $x \leq \frac{1}{5}$

3) $|2x-3| \leq 1$

$|2x-3| \leq 1 \rightarrow -1 \leq 2x-3 \leq 1$

$-1+3 \leq 2x-3 \leq 1+3 \quad \rightarrow \quad [2 \leq 2x \leq 4] \div 2 \rightarrow 1 \leq x \leq 2$

0.1 Inequalities

Ex: Solve for x the inequality $2x - 3 < 7$

$2x < 10$

$x < 5$

∴ the set of sol. $= \{x : x \in R , x < 5\}$

$= (-\infty, 5)$

Ex: Solve for x $3+7x \leq 2x - 9$

$7x - 2x \leq -9 - 3$

$5x \leq -12$

$x \leq -\frac{12}{5}$

∴ the set of sol. $= \{x : x \in R, x \leq -\frac{12}{5}\} = (-\infty , -\frac{12}{5}]$

Ex: Solve for x $7 \leq 2\text{-}5x < 9$

$5 \leq -5\,x < 7$

$-5 \geq 5\,x > -7$

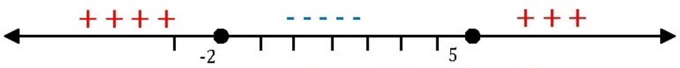
$-1 \geq x > -\frac{7}{5}$

∴ the set of sol. $= \{x : x \in R, -\frac{7}{5} < x \leq -1\} = (-\frac{7}{5} , -1]$

Ex: Solve for x $x^2 - 3x - 10 \geq 0$

$(x+2)(x-5) \geq 0$

equal to zero at $x = -2$ $x = 5$



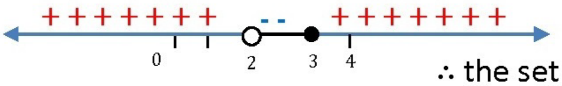
sign of (x+2) (x-5)

∴ set of sol. $= (-\infty, -2] \cup [5 , \infty)$

Ex: Solve for x $\frac{2x-5}{x-2} \leq 1$

$\frac{2x-5}{x-2} - 1 \leq 0$

$\frac{(2x-5)-(x-2)}{(x-2)} \leq 0 \implies \frac{x-3}{x-2} \leq 0$



∴ the set of sol. $= (2, 3]$

hussein_awadh@uomus.edu.iq

Ex: Solve for x the inequality $x^3 - 3x + 2 \leq 0$

$x = 1$ is a solution for the equation so $(x-1)$ is a factor.

$$x^3 - 3x + 2 \leq 0$$

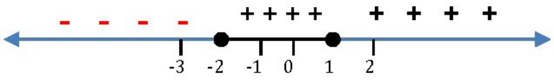
$$(x-1)(x^2 + x - 2) \leq 0$$

$$(x-1)(x-1)(x+2) \leq 0$$

equal to zero at $x=1$, $x=-2$

$$\begin{array}{r l}
 & x^2 + x - 2 \\
(x-1) \big) & \overline{x^3 - 3x + 2} \; - \\
 & \underline{\mp x^3 \pm x^2} \\
 & x^2 - 3x + 2 \\
 & \underline{\mp x^2 \pm x} \\
 & -2x + 2 \\
 & \underline{\pm 2x \mp 2} \\
 & 0 + 0
\end{array}$$

∴ the set of sol. $= (-\infty, -2]$



HW: Solve for x

1) $\frac{3x+1}{x-2} < 1$
2) $x^2 \leq 5$
3) $2 - 3x + x^2 \geq 0$
4) $\frac{1}{x+1} \geq \frac{3}{x-2}$
5) $x^3 - x^2 - x - 2 > 0$

Absolute Value

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

1) $|a| = \sqrt{a^2}$
2) $|a.b| = |a|\,|b|$
3) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4) $|a + b| \leq |a| + |b|$
5) If $|x| \leq a$ then $-a \leq x \leq a$
6) If $|x| \geq a$ either $x \geq a$ or $x \leq -a$

***ex*: solve $|x-3| = 4$**

either	Or
$(x-3) = 4$	$-(x-3) = 4$
$x = 7$	$-x = 1$
$x = 7$	$x = -1$

$\therefore$ set of sol. = $\{-1, 7\}$

***Ex:* solve for x $|x-3| < 4$**

$-4 < x-3 < 4$

$-1 < x < 7$

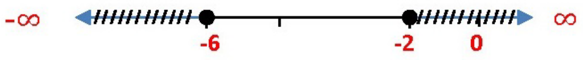
$\therefore$ set of sol. $= \{x : -1 < x < 7\} = (-1, 7)$

***Ex*: solve for x $|x+4| \geq 2$**

Either	Or
$x+4 \geq 2$	$x+4 \leq -2$
$x \geq -2$	$x \leq -6$

$\therefore$ set of sol. $= \{x : x \geq -2\} \cup \{x : x \leq -6\}$

$= (-\infty, -6] \cup [-2, \infty)$



***Ex*: solve for x $\frac{2}{|x+3|} < 1$**

$\frac{|x+3|}{2} > 1$

$|x+3| > 2$

Either	Or
$x+3 > 2$	$x+3 < -2$
$x > -1$	$x < -5$

$\therefore$ set of sol. $= \{x : x < -5\} \cup \{x : x > -1\}$

$= (-\infty, -5) \cup (-1, \infty)$

hussein_awadh@uomus.edu.iq

Ex: **solve for x** $|x+3| < |x-8|$

$$\sqrt{(x+3)^2} < \sqrt{(x-8)^2} \quad \text{using } |a| = \sqrt{a^2}$$

$$(x+3)^2 < (x-8)^2$$

$$x^2 + 6x + 9 < x^2 - 16x + 64$$

$$22x < 55$$

$$x < \frac{5}{2}$$

∴ set of sol. = $(-\infty , \frac{5}{2})$

HW: solve for x

1) $|3x| \leq |2x-5|$
2) $\left|\frac{3-2x}{1+x}\right| \leq 4$
3) $\frac{1}{|x-3|} - \frac{1}{|x+4|} \geq 0$
4) $\frac{1}{|x-4|} < \frac{1}{|x+7|}$
5) Solve $|x-3|^2 - 4\,|x-3| = 12$

Try to solve X value Below

1-

2-

3-

4-

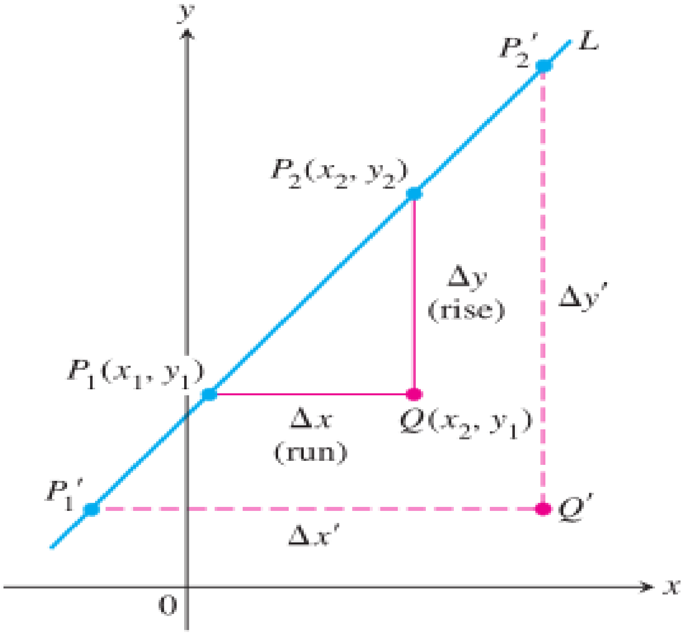
5-

Functions and their graphs:

Increments and Straight Lines:

When a particle moves from one point in the plane to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 and y increment changes from y_1 to y_2 in y then the increment in x and y respectively is:

$$\Delta x = x_2 - x_1 \qquad and \qquad \Delta y = y_2 - y_1$$



Slope:

Definition: the slope of the nonvertical line $p_1(x_1, y_1)$ *and* $p_2(x_2, y_2)$

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one-point $\boldsymbol{p_1(x_1, y_1)}$on it. If $\boldsymbol{p_1}$ $(\boldsymbol{x_1}, \boldsymbol{y_1})$ is any other point on L, then we can use the two points $p_1$and p to compute the slope,

$$\boldsymbol{m = \frac{y - y_1}{x - x_1}} \quad \textbf{or} \quad \boldsymbol{y = y_1 + m(x - x_1)}$$

This equation is called the point-slope equation of the line that passes through the point $\boldsymbol{p1}$ $(\boldsymbol{x1}, \boldsymbol{y1})$ and has slope $\boldsymbol{m}$.

Example: Write an equation for the line through the point (2, 3) with slope -3/2.
Sol:

$$\boldsymbol{y = y_1 + m(x - x_1)}$$

We substitute $x_1 = 2$ *and* $y_1 = 3$ into the point-slope equation and obtain

$$y = 3 - \frac{3}{2}(x - 2) = 3 - \frac{3}{2}\,x + 3 = -\frac{3}{2}\,x + 6$$

Example: Write an equation for the line through (-2,-1) and (3, 4).

Sol:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-1)}{3-(-2)} = \frac{4+1}{3+2} = \frac{5}{5} = 1$$

We can use this slope with either of the two given points in the point-slope equation:

$$y = y_1 + m(x - x_1) = -1 + 1(x+2) = x + 1$$

Tangent Line:

The tangent line to the curve at P is the line through P with this slope. Finding the tangent $y = f(x)$ to the curve at (x, y) by derive the function y with respect to x and then apply :

$$y = y_\circ + m(x - x_\circ)$$

Example: Find the slope of the curve and the tangent line of

$y = 1 + x^2 \ \ at\ (2,5)$.

The slope

$$m = \frac{dy}{dx} = 2x \quad at\ x = 2 \qquad m = 2 * 2 = 4 \quad .$$

Tangent line

$$y = y_\circ + m(x - x_\circ) = 5 + 4(x-2) = 5 + 4x - 8 = 4x - 3$$

Graphs of Functions:

If f is a function with domain D, its graph consists of the points in the cartesian plane whose coordinates are the input-output pairs for f.

Example: Graph the function $y = x^2$ over the interval [-2,2].

Sol:

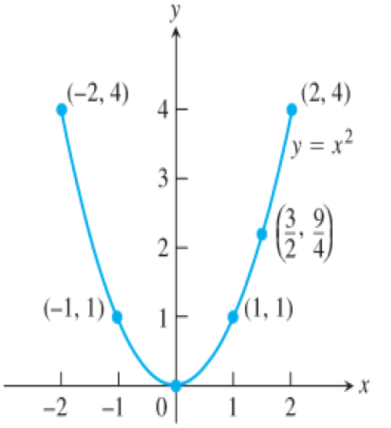
$we\ put\ x = 0 \rightarrow y = x^2 \rightarrow y = (0)^2 \rightarrow y = 0 \rightarrow (0,0)$

$x = 1 \rightarrow y = x^2 \rightarrow y = (1)^2 \rightarrow y = 1 \rightarrow (1,1)$

$x = -1 \rightarrow y = x^2 \rightarrow y = (-1)^2 \rightarrow y = 1 \rightarrow (-1,1)$

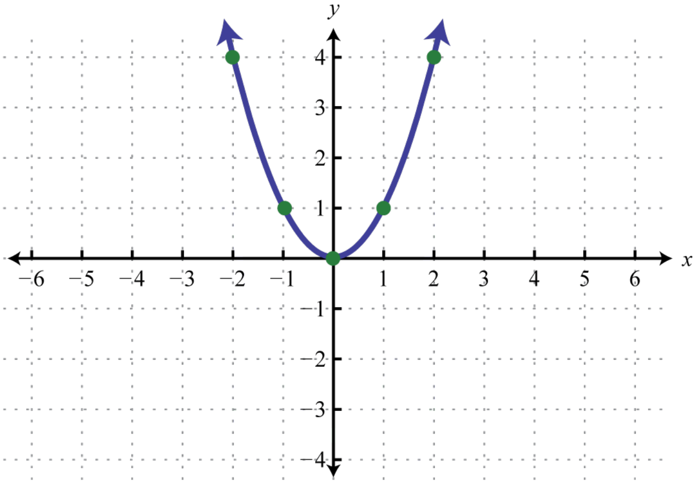
$x = -2 \rightarrow y = x^2 \rightarrow y = (-2)^2 \rightarrow y = 4 \rightarrow (-2,4)$

$x = 2 \rightarrow y = x^2 \rightarrow y = (2)^2 \rightarrow y = 4 \rightarrow (2,4)$



$f(x) = x^2$

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4



Example: Graph the function y =|x| → **{x>= 0 , x<0}**

when $y = x$

we put $x = 0 \rightarrow y = x \rightarrow y = 0 \rightarrow (0,0)$

$x = 1 \rightarrow y = x \rightarrow y = 1 \rightarrow (1,1)$

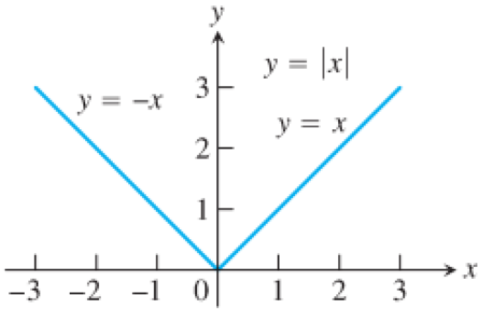
$x = 2 \rightarrow y = x \rightarrow y = 2 \rightarrow (2\,,2)$

and when $y = -x$

$x = 0 \rightarrow y = -x \rightarrow y = 0 \rightarrow (0,0)$

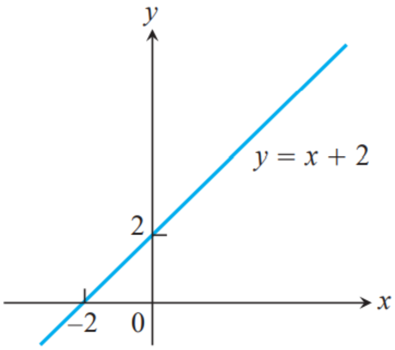
$x = -1 \rightarrow y = -x \rightarrow y = 1 \rightarrow (-1,1)$

$x = -2 \rightarrow y = -x \rightarrow y = 2 \rightarrow (-2,2)$



Example

The graph of the function $f(x)=x+2$ is the set of points with coordinates (x, y) for which $y=x+2$. Its graph is sketched below

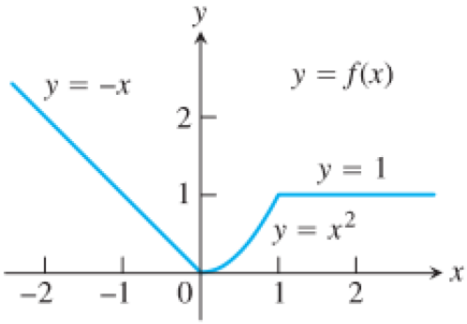


The graph of a function f is a useful picture of its behaviour. If (x, y) is a point on the graph, then y=f(x) is the height of the graph above the point x. The height may be positive or negative, depending on the sign of f(x).

Example: Sketch the graph for the function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Sol:



when $y = -x$

we put $x = 0 \rightarrow y = -x \rightarrow y = 0 \rightarrow (0,0)$

$x = -1 \rightarrow y = -x \rightarrow y = 1 \rightarrow (-1,1)$

$x = -2 \rightarrow y = -x \rightarrow y = 1 \rightarrow (-2,2)$

when $y = x^2$

$$x = 0 \rightarrow y = x^2 \rightarrow y = 0 \rightarrow (0,0)$$

and when $y = 1$

$x = 1 \rightarrow y = 1 \rightarrow (1,1)$

$x = 2 \rightarrow y = 1 \rightarrow (2,1)$

Even Functions and Odd Functions (Symmetry):

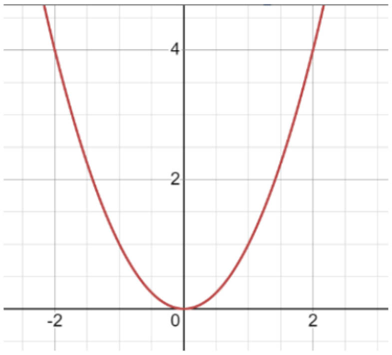
The graphs of even and odd functions have characteristic symmetry properties.

Definition: A function $y = f(x)$ *is an*:

-Even function of x *if* $f(-x) = f(x)$

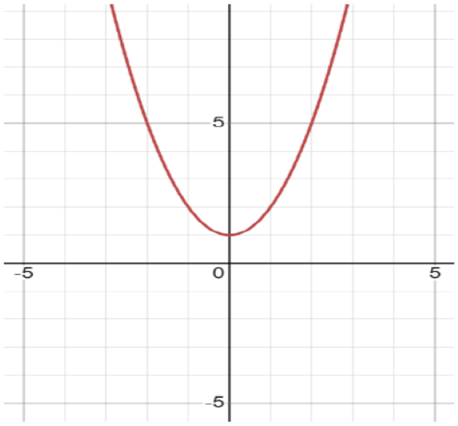
-Odd function of x *if* $f(-x) = -f(x)$

Example: $f(x) = x^2$ Even function: $(-\boldsymbol{x})^2 = \boldsymbol{x}^2$ for all x;



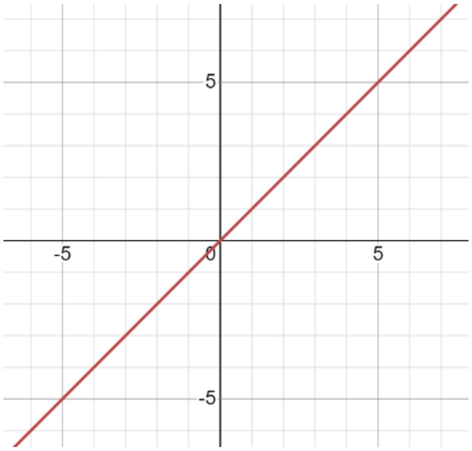
symmetry about y − axis.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ $for\ all\ x$;



symmetry about $y - axis$.

$f(x) = x$ Odd function $(-\boldsymbol{x}) = -\boldsymbol{x}$ $for\ all\ x$;



symmetry about the origin.

$f(x) = x + 1$ *Not odd*: $f(-x) = -x + 1,$

but $-f(x) = -x - 1.$ *The two are not equal.*

$$\boxed{\textbf{Not even}: (-x) + 1 \neq x + 1 \ \textbf{for all}\ x \neq 0}$$

Limit and Continuity:

When f (x) close to the number L as x close to the number a, we write

$$\boldsymbol{f(x) \to L \ \ as \ \ x \to a \qquad means: \ \lim_{x\to a} f(x) = L}$$

Example: Let $f\ (x) = 2x + 5\ evaluate\ f\ (x)\ at\ x = 1$

Sol:

$$\lim_{x\to a} f(x) = \lim_{x\to 1}\ (2x + 5) = 2 * 1 + 5 = 7$$

Example: If $f(x) = \frac{x^2-3x+2}{x-2}$ $, x \neq 2;\ \ find\ \lim_{x\to 2} f(x).$

Sol:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(\frac{x^2-3x+2}{x-2}) = \lim_{x\to 2}\frac{(x-2)(x-1)}{x-2} = \lim_{x\to 2}(x-1) = 2 - 1 = 1$$

Example: Evaluate the following limits if they exist.

1) $\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1}$ $\quad , x \neq -1,\ x \neq -2$

Sol:

$$\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1} * \frac{\sqrt{2+x}+1}{\sqrt{2+x}+1} = \lim_{x\to -1}\frac{2+x-1}{x+1(\sqrt{2+x}+1)}$$

$$\lim_{x\to -1}\frac{x+1}{x+1(\sqrt{2+x}+1)} = \lim_{x\to -1}\frac{1}{(\sqrt{2+x}+1)} = \frac{1}{(\sqrt{2-1}+1)} = \frac{1}{2}$$

2) $\lim\limits_{x\to 2}\frac{2-x}{2-\sqrt{2x}}$, $x \neq 2, x \geq 0$

Sol:

$$\lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}} * \frac{2+\sqrt{2x}}{2+\sqrt{2x}} = \lim_{x\to 2}\frac{2-x\left(2+\sqrt{2x}\right)}{4-2x}$$

$$\lim_{x\to 2}\frac{2-x\left(2+\sqrt{2x}\right)}{2(2-x)} = \lim_{x\to 2}\frac{\left(2+\sqrt{2x}\right)}{2} = \frac{\left(2+\sqrt{2*2}\right)}{2} = 2$$

The Limit Laws:

If L, M, C, and k are real numbers and $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} \mathrm{g(x)} = \mathrm{M}$

1. Sum Rule: $\lim\limits_{x\to c}\left(f(x)+g(x)\right) = L+M$

2. Difference Rule: $\lim\limits_{x\to c}\left(f(x)-g(x)\right) = L-M$

3. Constant Multiple Rule: $\lim\limits_{x\to c} k.f(x) = k.L$

4. Product Rule: $\lim\limits_{x\to c}\left(f(x).g(x)\right) = L.M$

5. Quotient Rule: $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

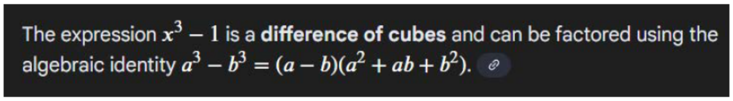
6. Power Rule: $\lim\limits_{x\to c}[f(x)]^n = L^n$

7. Root Rule: $\lim\limits_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L}$, n is a positive integer.

Example: Evaluate the following limits

1) $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$ $, x \neq 1$

Sol:



The expression $x^3 - 1$ is a **difference of cubes** and can be factored using the algebraic identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1} x^2+x+1 = 1^2+1+1 = 3$$

2) $\lim\limits_{h\to 0}\frac{1}{h}(\frac{1}{x+h}-\frac{1}{x}), h \neq 0$

Sol:

$$\lim_{h\to 0}[\frac{1}{h}(\frac{1}{x+h}-\frac{1}{x})] = \lim_{h\to 0}[\frac{1}{h}(\frac{x-x-h}{x(x+h)})] = \lim_{h\to 0}[\frac{1}{h}(\frac{-h}{x(x+h)}]$$

$$\lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x(x+0)} = -\frac{1}{x^2}$$

3) $\lim\limits_{x\to c}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim\limits_{x\to c}(x^4+x^2-1)}{\lim\limits_{x\to c}(x^2+5)} = \frac{c^4+c^2-1}{c^2+5}$

4) $\lim\limits_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim\limits_{x\to -2}(4x^2-3)} = \sqrt{4*(-2)^2-3} = \sqrt{16-3} = \sqrt{13}$

Limits of infinity:

We note when the limit of a function $f(x)$ exist and x approach at infinity, we write:

$\lim\limits_{x\to\infty} f(x) = L$ For positive values of x.

$\lim\limits_{x\to -\infty} f(x) = L$ For negative values of x.

Some obvious (clear) limits:

1- If k is constant, then $\lim\limits_{x\to\infty} k = k$

2- $\lim\limits_{x\to\infty} \frac{1}{x} = 0, \; \lim\limits_{x\to-\infty} \frac{1}{x} = \frac{1}{\infty} = 0$

3- $\lim\limits_{x\to 0} \frac{1}{x} = \infty$

Example: Find the following limits

1- $\lim\limits_{x\to\infty} \frac{x}{2x+3} = \lim\limits_{x\to\infty} \frac{1}{2+\frac{3}{x}} = \frac{1}{2+0} = \frac{1}{2}$

2- $\lim\limits_{x\to\infty} \frac{2x^2+3x+5}{5x^2-4x+1} = \lim\limits_{x\to\infty} \frac{2+\frac{3}{x}+\frac{5}{x^2}}{5-\frac{4}{x}+\frac{1}{x^2}} = \frac{2}{5}$

3- $\lim\limits_{x\to\infty} \frac{2x^2+1}{3x^2-2x^2+5x-2} = \lim\limits_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^3}}{3-\frac{2}{x}+\frac{5}{x^2}-\frac{2}{x^3}} = 0$

4- $\lim\limits_{x\to\infty} \frac{2x^2+1}{3x^2-2x^2+5x-2} = \lim\limits_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^3}}{3-\frac{2}{x}+\frac{5}{x^2}-\frac{2}{x^3}} = 0$

5- $\lim\limits_{x\to\infty} \left[\left(\sqrt{x^2+1}\right) - x * \frac{\left(\sqrt{x^2+1}\right)+x}{\left(\sqrt{x^2+1}\right)+x}\right] = \lim\limits_{x\to\infty} \frac{x^2+1-x^2}{\left(\sqrt{x^2+1}\right)+x}$

$$\lim_{x\to\infty} \frac{1}{\left(\sqrt{x^2+1}\right)+x} = \frac{\frac{1}{x^2}}{\left(\sqrt{x^2+1}\right)+x} = 0$$

Home Work

1- $\lim\limits_{x\to\infty} \frac{\sqrt{x^2-3x-2}}{2x+4}$

2- $\lim\limits_{x\to\infty} \frac{\sqrt{9x^6-x}}{x^3+6}$

3- $\lim\limits_{x\to\infty} \frac{\sqrt{9x^2+2x-5}}{3x}$

Continuous Function:

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions

1- $f(c)\ is\ exists.$

2- $\lim_{x \to c} f(x) = exists.$

3- $\lim_{x \to c} f(c) = c$

Example:

1) $f(x) = \frac{1}{x}$ is not continuous for all except $x = 0$

2) $f(x) = \frac{x+3}{(x-5)(x+2)}$ is dicontiunuos at $x = 5\ \ and\ x = -2$

3) $f(x) = \frac{\sin x}{x}$ is dicontiunuos at $x = 0$

4) $f(x) = \frac{x^2+x-6}{x^2-4}$ is dicontiunuos at $x = \pm 2$